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Rotating strings and D2-branes in type IIA reduction of M-theory on G_2 manifold and their semiclassical limits

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Abstract: We consider rotating strings and D2-branes on type IIA background, which arises as dimensional reduction of M-theory on manifold of G_2 holonomy, dual to $\mathcal{N} = 1$ gauge theory in four dimensions. We obtain exact solutions and explicit expressions for the conserved charges. By taking the semiclassical limit, we show that the rotating strings can reproduce only one type of semiclassical behavior, exhibited by rotating M2-branes on G_2 manifolds. Our further investigation leads to the conclusion that the rotating D2-branes reproduce two types of the semiclassical energy-charge relations known for membranes in eleven dimensions.

Keywords: Long strings, D-branes, Gauge-gravity correspondence.

Contents

1. Introduction

In the recent years, an essential progress has been achieved in understanding the semiclassical limit of the string/gauge theory duality [1]. This initiated also an interest in the investigation of the M-theory lift of this semiclassical correspondence and in particular, in obtaining new membrane solutions in curved space-times and finding relations between their energy and the other conserved charges [2]-[11]. So far, such relations have been obtained for the following target spaces: $AdS_p \times S^q$ [2, 3, 5, 8]-[10], $AdS_4 \times Q^{1,1,1}$ [5], warped $AdS_5 \times M^6$ [5], 11-dimensional AdS-black hole [5], and manifolds of G_2 holonomy [5, 11]. In [5], various rotating membrane configurations on different G_2 holonomy backgrounds have been studied systematically. In the semiclassical limit (large conserved charges), the following relations between the energy and the corresponding charge K have been obtained: $E \sim K^{1/2}$, $E \sim K^{2/3}$, $E - K \sim K^{1/3}$, $E - K \sim \ln K$. In [11], rotating membranes on a manifold with exactly known metric of G_2 holonomy [12] have been considered. The above energy-charge relations, except the last one, have been reproduced and generalized for the case of more than one conserved charges. Moreover, examples of more complicated dependence of the energy on the charges have been found. The most general cases considered, lead to algebraic equations of third or even forth order for the E^2 as function of up to five conserved momenta.

JHEP08(2006)029

It seems to us that an interesting task is to check if rotating strings in type IIA theory in ten dimensions, can reproduce the energy-charge relations obtained in [5] and [11] for rotating M2-branes.

In this paper, we consider rotating strings on type IIA background, which arises as dimensional reduction of M-theory on the manifold of G_2 holonomy, discovered in [12]. By taking the semiclassical limit, we obtain that the rotating strings can reproduce only one type of semiclassical behavior, exhibited by rotating M2-branes on G_2 manifolds. Namely, $E \sim K^{1/2}$ and generalizations thereof. Our further investigation shows that the rotating D2-branes reproduce two types of the semiclassical energy-charge relations known for membranes in M-theory. These are generalizations of the dependencies $E \sim K^{1/2}$ and $E \sim K^{2/3}$.

The paper is organized as follows. In section 2, we describe the type IIA background, which we will use. In section 3, we settle the framework, which we will work in. In section 4, we obtain three types of rotating string solutions and explicit expressions for the corresponding conserved charges. Then, we take the semiclassical limit and derive different energy-charge relations. In section 5, the same is done for rotating D2-branes. Section 6 is devoted to our concluding remarks.

2. The type IIA background

The type IIA background, in which we will search for rotating string and D2-brane solutions, has the form [12]

$$\begin{aligned}
ds^2_{10} &= r_0^{1/2} C\left\{-(dx^0)^2 + \delta_{IJ}dx^I dx^J + A^2\left[(g^1)^2+(g^2)^2\right]\right.\\
&\left.+\ B^2\left[(g^3)^2+(g^4)^2\right] + D^2(g^5)^2\right\} + r_0^{1/2}\frac{dr^2}{C}, \quad (I,J=1,2,3), \quad r_0 = const,\\
e^{\Phi} &= r_0^{3/4}C^{3/2}, \quad F_2 = \sin\theta_1 d\phi_1\wedge d\theta_1 - \sin\theta_2 d\phi_2\wedge d\theta_2. \qquad (2.1)
\end{aligned}$$

Here, g^1,...,g^5 are given by

$$\begin{aligned}
g^1 &= -\sin\theta_1 d\phi_1 - \cos\psi_1\sin\theta_2 d\phi_2 + \sin\psi_1 d\theta_2,\\
g^2 &= d\theta_1 - \sin\psi_1\sin\theta_2 d\phi_2 - \cos\psi_1 d\theta_2,\\
g^3 &= -\sin\theta_1 d\phi_1 + \cos\psi_1\sin\theta_2 d\phi_2 - \sin\psi_1 d\theta_2,\\
g^4 &= d\theta_1 + \sin\psi_1\sin\theta_2 d\phi_2 + \cos\psi_1 d\theta_2,\\
g^5 &= d\psi_1 + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2,
\end{aligned}$$

and the functions A, B, C and D depend on the radial coordinate r only:

$$\begin{aligned}
A &= \frac{1}{\sqrt{12}}\sqrt{(r-3r_0/2)(r+9r_0/2)}, \quad B = \frac{1}{\sqrt{12}}\sqrt{(r+3r_0/2)(r-9r_0/2)},\\
C &= \sqrt{\frac{(r-9r_0/2)(r+9r_0/2)}{(r-3r_0/2)(r+3r_0/2)}}, \quad D = r/3. \qquad (2.2)
\end{aligned}$$

In (2.1), Φ and F_2 are the Type IIA dilaton and the field strength of the Ramond-Ramond one-form gauge field respectively.

JHEP08(2006)029

The above ten dimensional background arises as dimensional reduction of the following solution of the eleven dimensional supergravity [12]

$$l_{11}^{-2}ds_{11}^2 = -(dx^0)^2 + \delta_{IJ}dx^I dx^J + ds_7^2, \tag{2.3}$$
$$ds_7^2 = dr^2/C^2 + A^2\left[(g^1)^2 + (g^2)^2\right] + B^2\left[(g^3)^2 + (g^4)^2\right] + D^2(g^5)^2 + r_0\, C^2(g^6)^2,$$

where l_{11} is the eleven dimensional Planck length and

$$g^6 = d\psi_2 + \cos\theta_1 d\phi_1 - \cos\theta_2 d\phi_2.$$

The type IIA solution (2.1) describes a D6-brane wrapping the $\mathbf{S}^3$ in the deformed conifold geometry. For $r \to \infty$, the metric becomes that of a singular conifold, the dilaton is constant, and the flux is through the $\mathbf{S}^2$ surrounding the wrapped D6-brane. For $r - 9r_0/2 = \epsilon \to 0$, the string coupling e^Φ goes to zero like $\epsilon^{3/4}$, whereas the curvature blows up as $\epsilon^{-3/2}$ just like in the near horizon region of a flat D6-brane. This means that classical supergravity is valid for sufficiently large radius. However, the singularity in the interior is the same as the one of flat D6 branes, as expected. On the other hand, the dilaton continuously decreases from a finite value at infinity to zero, so that for small r_0 classical string theory is valid everywhere. As explained in [12], the global geometry is that of a warped product of flat Minkowski space and a non-compact space, Y_6, which for large radius is simply the conifold since the backreaction of the wrapped D6 brane becomes less and less important. However, in the interior, the backreaction induces changes on Y_6 away from the conifold geometry. For $r \to 9r_0/2$, the $\mathbf{S}^2$ shrinks to zero size, whereas an $\mathbf{S}^3$ of finite size remains. This behavior is similar to that of the deformed conifold but the two metrics are different.

3. The set-up

The ten dimensional background, described in the previous section, does not depend on part of the target space coordinates x^M, $M = 0, 1, \ldots, 9$. We denote them by x^μ and the remaining ones by x^a: $x^M = (x^\mu, x^a)$. Further on, we will use the following ansatz for the string and D2-brane embedding coordinates $x^M = X^M(\xi^m)$

$$X^\mu(\xi^m) = \Lambda^\mu_m \xi^m, \qquad X^a(\xi^m) = Z^a(\xi^p), \qquad \xi^m = (\xi^0, \ldots, \xi^p), \tag{3.1}$$

where Λ^μ_m are constants, $\xi^p = \xi^1$ for the string and $\xi^p = \xi^2$ for the D2-brane.

3.1 Rotating strings

In our further considerations, we will use the Polyakov action for strings embedded in curved space-time with metric tensor $g_{MN}(x)$, interacting with a background 2-form gauge field $b_{MN}(x)$ via Wess-Zumino term

$$S^P = -\frac{T}{2}\int d^2\xi \left(\sqrt{-\gamma}\gamma^{mn}G_{mn} - \varepsilon^{mn}B_{mn}\right), \tag{3.2}$$
$$\xi^m = (\xi^0, \xi^1), \qquad m, n = (0, 1),$$

JHEP08(2006)029

where

$$G_{mn} = \partial_m X^M \partial_n X^N g_{MN}, \quad B_{mn} = \partial_m X^M \partial_n X^N b_{MN}, \quad (\partial_m = \partial/\partial\xi^m),$$

are the fields induced on the string worldsheet, γ is the determinant of the auxiliary worldsheet metric γ_{mn}, γ^{mn} is its inverse, and $T = 1/2\pi\alpha'$ is the string tension.

For our background (2.1), the action (3.2) reduces to

$$S^P = \int d^2\xi \mathcal{L}^P, \quad \mathcal{L}^P = -\frac{T}{2}\sqrt{-\gamma}\gamma^{mn}G_{mn}. \tag{3.3}$$

The equations of motion for X^M following from (3.3) are:

$$-g_{LK}\left[\partial_m\left(\sqrt{-\gamma}\gamma^{mn}\partial_n X^K\right) + \sqrt{-\gamma}\gamma^{mn}\Gamma^K_{MN}\partial_m X^M \partial_n X^N\right] = 0, \tag{3.4}$$

where

$$\Gamma_{L,MN} = g_{LK}\Gamma^K_{MN} = \frac{1}{2}\left(\partial_M g_{NL} + \partial_N g_{ML} - \partial_L g_{MN}\right),$$

are the components of the symmetric connection corresponding to the metric g_{MN}. The constraints are obtained by varying the action (3.3) with respect to γ_{mn}:

$$\delta_{\gamma_{mn}} S^P = 0 \Rightarrow \left(\gamma^{kl}\gamma^{mn} - 2\gamma^{km}\gamma^{ln}\right) G_{mn} = 0. \tag{3.5}$$

Further on, we will work in conformal gauge $\gamma^{mn} = \eta^{mn} = diag(-1,1)$, in which the equations of motion (3.4) and constraints (3.5) simplify to

$$g_{LK}\eta^{mn}\left(\partial_m\partial_n X^K + \Gamma^K_{MN}\partial_m X^M \partial_n X^N\right) = 0. \tag{3.6}$$

$$G_{00} + G_{11} = 0, \tag{3.7}$$

$$G_{01} = 0. \tag{3.8}$$

Taking into account the ansatz (3.1), one obtains that the metric induced on the string worldsheet is given by (the prime is used for $d/d\xi^1$)

$$\begin{aligned} G_{00} &= \Lambda_0^\mu \Lambda_0^\nu g_{\mu\nu}, \quad G_{11} = g_{ab}Z'^a Z'^b + 2\Lambda_1^\mu g_{\mu a}Z'^a + \Lambda_1^\mu\Lambda_1^\nu g_{\mu\nu}, \\ G_{01} &= \Lambda_0^\mu\left(g_{\mu a}Z'^a + \Lambda_1^\nu g_{\mu\nu}\right). \end{aligned}$$

The Lagrangian density in the action (3.3) reduces to

$$\mathcal{L}_s^A(\xi^1) = -\frac{T}{2}\left(g_{ab}Z'^a Z'^b + 2\Lambda_1^\mu g_{\mu a}Z'^a + \eta^{mn}\Lambda_m^\mu\Lambda_n^\nu g_{\mu\nu}\right). \tag{3.9}$$

$\mathcal{L}_s^A$ does not depend on X^μ, so the conjugated momenta

$$P_\mu = T\Lambda_0^\nu \int d\xi^1 g_{\mu\nu} \tag{3.10}$$

are conserved, i.e. they do not depend on the proper time ξ^0.

Let us introduce the density

$$\mathcal{P}_M \equiv \frac{\partial \mathcal{L}^P}{\partial(\partial_1 X^M)} = -T\sqrt{-\gamma}\gamma^{1n} g_{MN}\partial_n X^N = -T\left(g_{Mb}Z'^b + \Lambda_1^\nu g_{M\nu}\right). \tag{3.11}$$

In terms of $\mathcal{P}_M$, the equations of motion (3.6) read

$$\left[\mathcal{P}_\mu(\xi^1)\right]' = 0, \tag{3.12}$$

$$(\mathcal{P}_a)' - \frac{\partial \mathcal{L}_s^A}{\partial Z^a} = 0. \tag{3.13}$$

The equations (3.12) mean that $\mathcal{P}_\mu$ are constants of the motion: $\mathcal{P}_\mu = constants$. The remaining equations (3.13) may be rewritten as

$$g_{ab}Z''^b + \Gamma_{a,bc}Z'^b Z'^c = \frac{1}{2}\partial_a \mathcal{U} + 2\partial_{[a}\mathcal{A}_{b]}Z'^b, \tag{3.14}$$
$$\partial_{[a}\mathcal{A}_{b]} = \frac{1}{2}\left(\partial_a \mathcal{A}_b - \partial_b \mathcal{A}_a\right).$$

In (3.14), an effective scalar potential $\mathcal{U}$ and an effective 1-form gauge field $\mathcal{A}_a$ appeared. They are given by

$$\mathcal{U} = \eta^{mn}\Lambda_m^\mu \Lambda_n^\nu g_{\mu\nu} + \frac{2\Lambda_1^\mu \mathcal{P}_\mu}{T}, \quad \mathcal{A}_a = \Lambda_1^\mu g_{a\mu}.$$

The constraints (3.7), (3.8) take the form

$$g_{ab}Z'^a Z'^b = \mathcal{U}, \quad \Lambda_0^\mu\left(g_{\mu a}Z'^a + \Lambda_1^\nu g_{\mu\nu}\right) = 0. \tag{3.15}$$

Here, we are interested in obtaining rotating string solutions for which the conditions $\mathcal{P}_\mu = constants$ and the second constraint in (3.15) are identically satisfied by appropriate choice of the embedding parameters Λ_m^μ. Then, the problem reduces to solving the equations of motion (3.14) and the first constraint in (3.15). We further restrict ourselves to the simplest case, when the embedding is such that the background seen by the string depends only on the radial coordinate r. In this case, the solution is [13]

$$\xi^1(r) = \int_{r_{min}}^{r}\left(\frac{g_{rr}}{\mathcal{U}}\right)^{1/2} dt. \tag{3.16}$$

On the solution (3.16), the conserved generalized momenta (3.10) take the form

$$P_\mu = 2T\Lambda_0^\nu \int_{r_{min}}^{r_{max}} g_{\mu\nu}\left(\frac{g_{rr}}{\mathcal{U}}\right)^{1/2} dt. \tag{3.17}$$

3.2 Rotating D2-branes

The Dirac-Born-Infeld type action for D2-brane in ten dimensional space-time with metric tensor $g_{MN}(x)$, interacting with a background 3-form Ramond-Ramond gauge field $c_{MNP}(x)$ via Wess-Zumino term, can be written in string frame as

$$S^{DBI} = -T_{D2}\int d^3\xi\Big\{e^{-\Phi}\sqrt{-\det\left(G_{mn} + B_{mn} + 2\pi\alpha' F_{mn}\right)} \tag{3.18}$$
$$- \frac{\varepsilon^{m_1 m_2 m_3}}{3!}\partial_{m_1}X^{M_1}\partial_{m_2}X^{M_2}\partial_{m_3}X^{M_3}c_{M_1 M_2 M_3}\Big\}.$$

Here, T_{D2} is the D2-brane tension, G_{mn}, B_{mn} and Φ are the pullbacks of the background metric, antisymmetric tensor and dilaton to the D2-brane worldvolume, while F_{mn} is the field strength of the worldvolume U(1) gauge field A_m: $F_{mn} = 2\partial_{[m}A_{n]}$. For our background, (3.18) reduces to [1]

$$S^{DBI} = -T_{D2}\int d^3\xi e^{-\Phi}\sqrt{-\det G_{mn}},$$

which is classically equivalent to the following action [14]

$$S_{D2} = \int d^3\xi\mathcal{L}_{D2} = \int d^3\xi\frac{e^{-\Phi}}{4\lambda^0}\Big[G_{00} - 2\lambda^i G_{0i} + \lambda^i\lambda^j G_{ij} - \left(2\lambda^0 T_{D2}\right)^2\det G_{ij}\Big], \tag{3.19}$$

where $\lambda^m = (\lambda^0,\ \lambda^i)$, $(i, j = 1, 2)$ are Lagrange multipliers, which equations of motion generate the *independent* constraints

$$G_{00} - 2\lambda^j G_{0j} + \lambda^i\lambda^j G_{ij} + \left(2\lambda^0 T_{D2}\right)^2\det G_{ij} = 0, \tag{3.20}$$

$$G_{0j} - \lambda^i G_{ij} = 0. \tag{3.21}$$

Further on, we will use the action (3.19) because it does not contain square root opposite to the DBI type action (3.18), thus avoiding the introduction of additional nonlinearities in the equations of motion.

The equations of motion for X^M following from (3.19), in the worldvolume gauge $\lambda^m = consants$, are ($\mathbf{G} \equiv \det G_{ij}$)

$$\begin{aligned}
&g_{MN}\left[\left(\partial_0 - \lambda^i\partial_i\right)\left(\partial_0 - \lambda^j\partial_j\right)X^N - \left(2\lambda^0 T_{D2}\right)^2\partial_i\left(\mathbf{G}G^{ij}\partial_j X^N\right)\right] \\
&+ \left[\Gamma_{M,NK} - \left(g_{MK}\partial_N\Phi - \frac{1}{2}g_{NK}\partial_M\Phi\right)\right]\left(\partial_0 - \lambda^i\partial_i\right)X^N\left(\partial_0 - \lambda^j\partial_j\right)X^K \\
&- \left(2\lambda^0 T_{D2}\right)^2\mathbf{G}\left[\left(\Gamma_{M,NK} - g_{MK}\partial_N\Phi\right)G^{ij}\partial_i X^N\partial_j X^K + \frac{1}{2}\partial_M\Phi\right] = 0.
\end{aligned} \tag{3.22}$$

In practice, it turns out that using the diagonal gauge $\lambda^i = 0$ simplify the considerations a lot [9]. That is why, we restrict ourselves namely to this gauge from now on. In this case, (3.19), (3.20), (3.21) and (3.22) reduce to

$$S^{gf}_{D2} = \int d^3\xi\mathcal{L}^{gf}_{D2} = \int d^3\xi\frac{e^{-\Phi}}{4\lambda^0}\Big[G_{00} - \left(2\lambda^0 T_{D2}\right)^2\mathbf{G}\Big], \tag{3.23}$$

$$G_{00} + \left(2\lambda^0 T_{D2}\right)^2\mathbf{G} = 0, \tag{3.24}$$

$$G_{0i} = 0, \tag{3.25}$$

$$\begin{aligned}
&g_{MN}\left[\partial_0^2 X^N - \left(2\lambda^0 T_{D2}\right)^2\partial_i\left(\mathbf{G}G^{ij}\partial_j X^N\right)\right] \\
&+ \left[\Gamma_{M,NK} - \left(g_{MK}\partial_N\Phi - \frac{1}{2}g_{NK}\partial_M\Phi\right)\right]\partial_0 X^N\partial_0 X^K \\
&- \left(2\lambda^0 T_{D2}\right)^2\mathbf{G}\left[\left(\Gamma_{M,NK} - g_{MK}\partial_N\Phi\right)G^{ij}\partial_i X^N\partial_j X^K + \frac{1}{2}\partial_M\Phi\right] = 0.
\end{aligned} \tag{3.26}$$

[1]For $A_m = \partial_m f$.

JHEP08(2006)029

Taking into account the ansatz (3.1), one obtains that the metric induced on the D2-brane worldvolume is given by (the prime is used for $d/d\xi^2$)

$$
\begin{aligned}
&G_{00} = \Lambda_0^\mu \Lambda_0^\nu g_{\mu\nu}, \quad G_{11} = \Lambda_1^\mu \Lambda_1^\nu g_{\mu\nu}, \quad G_{22} = g_{ab} Z'^a Z'^b + 2\Lambda_2^\mu g_{\mu a} Z'^a + \Lambda_2^\mu \Lambda_2^\nu g_{\mu\nu}, \\
&G_{01} = \Lambda_0^\mu \Lambda_1^\nu g_{\mu\nu}, \quad G_{02} = \Lambda_0^\mu \left(g_{\mu a} Z'^a + \Lambda_2^\nu g_{\mu\nu} \right), \quad G_{12} = \Lambda_1^\mu \left(g_{\mu a} Z'^a + \Lambda_2^\nu g_{\mu\nu} \right).
\end{aligned}
$$

Correspondingly, the Lagrangian density in the action (3.23) reduces to

$$
\mathcal{L}^A(\xi^2) = \frac{1}{4\lambda^0} \left(\tilde{K}_{ab} Z'^a Z'^b + 2\tilde{A}_a Z'^a - \tilde{V} \right), \tag{3.27}
$$

where

$$
\begin{aligned}
\tilde{K}_{ab} &= -\left(2\lambda^0 T_{D2}\right)^2 \Lambda_1^\mu \Lambda_1^\nu \; \left(g_{ab} g_{\mu\nu} - g_{a\mu} g_{b\nu} \right) e^{-\Phi}, \\
\tilde{A}_a &= \left(2\lambda^0 T_{D2}\right)^2 \Lambda_1^\mu \Lambda_1^\nu \Lambda_2^\rho \left(g_{a\mu} g_{\nu\rho} - g_{a\rho} g_{\mu\nu} \right) e^{-\Phi}, \\
\tilde{V} &= \left[-\Lambda_0^\mu \Lambda_0^\nu g_{\mu\nu} + \left(2\lambda^0 T_{D2}\right)^2 \Lambda_1^\mu \Lambda_1^\nu \Lambda_2^\rho \Lambda_2^\lambda \left(g_{\mu\nu} g_{\rho\lambda} - g_{\mu\rho} g_{\nu\lambda} \right) \right] e^{-\Phi}.
\end{aligned}
$$

As far as $\mathcal{L}^A$ does not depend on X^μ, the momenta

$$
P_\mu = \frac{\Lambda_0^\nu}{2\lambda^0} \int\int d\xi^1 d\xi^2 g_{\mu\nu} e^{-\Phi} \tag{3.28}
$$

are conserved.

If we introduce the densities

$$
\mathcal{P}_M^i = \frac{\partial \mathcal{L}_{D2}^{gf}}{\partial \left(\partial_i X^M \right)},
$$

the equations of motion (3.26) acquire the form

$$
\left[\mathcal{P}_\mu^2(\xi^2) \right]' = 0, \tag{3.29}
$$

$$
\left(\mathcal{P}_a^2 \right)' - \frac{\partial \mathcal{L}^A}{\partial Z^a} = 0. \tag{3.30}
$$

The equations (3.29) just state that $\mathcal{P}_\mu^2$ are constants of the motion:

$$
\mathcal{P}_\mu^2 = 2\lambda^0 T_{D2}^2 e^{-\Phi} \Lambda_1^\nu \Lambda_1^\rho \left[\left(g_{\mu\nu} g_{\rho a} - g_{\mu a} g_{\nu\rho} \right) Z'^a + \Lambda_2^\lambda \left(g_{\mu\nu} g_{\rho\lambda} - g_{\mu\lambda} g_{\nu\rho} \right) \right] = constants. \tag{3.31}
$$

In the case under consideration, this is possible only for $\mathcal{P}_\mu^2 = 0$. The remaining equations (3.30) may be rewritten as

$$
\tilde{K}_{ab} Z''^b + \tilde{\Gamma}_{a,bc} Z'^b Z'^c - 2\partial_{[a} \tilde{A}_{b]} Z'^b + \frac{1}{2} \partial_a \tilde{V} = 0, \tag{3.32}
$$

where

$$
\tilde{\Gamma}_{a,bc} = \frac{1}{2} \left(\partial_b \tilde{K}_{ca} + \partial_c \tilde{K}_{ba} - \partial_a \tilde{K}_{bc} \right).
$$

The constraints (3.24) and (3.25) take the form

$$\tilde{K}_{ab}Z'^{a}Z'^{b} + \tilde{V} = 0, \tag{3.33}$$

$$\Lambda_0^{\mu}\Lambda_1^{\nu}g_{\mu\nu} = 0, \tag{3.34}$$

$$\Lambda_0^{\mu}\left(g_{\mu a}Z'^{a} + \Lambda_2^{\nu}g_{\mu\nu}\right) = 0. \tag{3.35}$$

We will search for D2-brane solutions for which the conditions (3.34), (3.35) and $\mathcal{P}_{\mu}^{2} = 0$ are identically satisfied due to appropriate choice of the embedding parameters Λ_m^{μ}. Then, the investigation of the D2-brane dynamics reduces to the problem of solving the equations of motion (3.32) and the remaining constraint (3.33). In this article, we restrict ourselves to the simplest case, when the embedding is such that the background seen by the D2-brane depends on the radial coordinate r only. Then, the constraint (3.33) is first integral of the equation of motion (3.32) for $Z^a(\xi^2) = r(\xi^2)$, and the solution is given by

$$\xi^2\left(r\right) = \int_{r_{min}}^{r}\left(-\frac{\tilde{K}_{rr}}{\tilde{V}}\right)^{1/2}dt. \tag{3.36}$$

On the solution (3.36), the conserved generalized momenta (3.28) take the form

$$P_{\mu} = \frac{\pi\Lambda_0^{\nu}}{\lambda^0}\int_{r_{min}}^{r_{max}} g_{\mu\nu}\left(-\frac{\tilde{K}_{rr}}{\tilde{V}}\right)^{1/2}e^{-\Phi}dt. \tag{3.37}$$

4. Rotating string solutions, conserved charges and their semiclassical limits

As we already mentioned in the previous section, we are interested here in obtaining rotating string solutions, for which the embedding is such that the background seen by the string depends only on the radial coordinate r. This leads to the following three cases[2]

1. ψ_1, ϕ_1, ϕ_2 **fixed to** ψ_1^0, ϕ_1^0, ϕ_2^0

$$\begin{aligned} ds^2 =& \, r_0^{1/2}\left\{C\left[-(dx^0)^2 + \delta_{IJ}dx^I dx^J + \left(A^2+B^2\right)\left(d\theta_1^2 + d\theta_2^2\right)\right.\right. \\ &- \left.\left. 2\left(A^2 - B^2\right)\cos\psi_1^0 d\theta_1 d\theta_2\right] + \frac{dr^2}{C}\right\}. \end{aligned} \tag{4.1}$$

2. ψ_1, ϕ_1, θ_2 **fixed to** ψ_1^0, ϕ_1^0, θ_2^0

$$\begin{aligned} ds^2 =& \, r_0^{1/2}\left\{C\left\{-(dx^0)^2 + \delta_{IJ}dx^I dx^J + \left(A^2+B^2\right)d\theta_1^2\right.\right. \\ &+ \left[\left(A^2+B^2\right)\sin^2\theta_2^0 + D^2\cos^2\theta_2^0\right]d\phi_2^2 \\ &- \left.\left. 2\left(A^2 - B^2\right)\sin\psi_1^0\sin\theta_2^0 d\theta_1 d\phi_2\right\} + \frac{dr^2}{C}\right\}. \end{aligned} \tag{4.2}$$

[2] For all of them $F_2 = 0$.

3. ψ_1, θ_1, θ_2 fixed to ψ_1^0, θ_1^0, θ_2^0

$$\begin{aligned} ds^2 = &\, r_0^{1/2}\left\{C\left\{-(dx^0)^2+\delta_{IJ}dx^I dx^J+\left[\left(A^2+B^2\right)\sin^2\theta_1^0+D^2\cos^2\theta_1^0\right]d\phi_1^2\right.\right.\\ &+\left[\left(A^2+B^2\right)\sin^2\theta_2^0+D^2\cos^2\theta_2^0\right]d\phi_2^2 \\ &\left.\left.+\,2\left[\left(A^2-B^2\right)\cos\psi_1^0\sin\theta_1^0\sin\theta_2^0+D^2\cos\theta_1^0\cos\theta_2^0\right]d\phi_1 d\phi_2\right\}+\frac{dr^2}{C}\right\}. \end{aligned} \tag{4.3}$$

There are also other possibilities, but they lead to the same type of metrics with respect to other coordinates.

Let us begin with considering string moving in the background (4.1). In this case, the most general ansatz of the type (3.1), which ensures that the conditions $\mathcal{P}_\mu = 0$ and the second constraint in (3.15) are identically satisfied is

$$X^0=\Lambda_0^0\xi^0,\quad X^I=\Lambda_0^I\xi^0,\quad r=r(\xi^1),\quad \theta_1=\Lambda_0^{\theta_1}\xi^0,\quad \theta_2=\Lambda_0^{\theta_2}\xi^0. \tag{4.4}$$

It corresponds to string extended in the radial direction r, and rotating in the planes given by the angles θ_1 and θ_2 with angular momenta P_{θ_1} and P_{θ_2}. At the same time, the string moves along x^0-coordinate with constant energy E, and along x^I with constant momenta P_I.

From the first constraint in (3.15),

$$g_{rr}r'^2-\mathcal{U}=\frac{r_0^{1/2}}{C}r'^2-r_0^{1/2}C\left(v_0^2-\Lambda_-^2A^2-\Lambda_+^2B^2\right)=0,$$

where

$$\begin{aligned} v_0^2 &= \left(\Lambda_0^0\right)^2-\delta_{IJ}\Lambda_0^I\Lambda_0^J=\left(\Lambda_0^0\right)^2-\mathbf{\Lambda}_0^2, \\ \Lambda_\pm^2 &= \left(\Lambda_0^{\theta_1}\right)^2+\left(\Lambda_0^{\theta_2}\right)^2\pm 2\Lambda_0^{\theta_1}\Lambda_0^{\theta_2}\cos\psi_1^0, \end{aligned} \tag{4.5}$$

one obtains the turning points of the effective one-dimensional periodic motion by solving the equation $r'=0$. In the case under consideration, the result is[3]

$$\begin{aligned} &r_{min}=9r_0/2\equiv 3l,\qquad r_{max}=r_1=l\left[2\sqrt{\frac{k^2+3}{4}+\frac{3v_0^2}{l^2\left(\Lambda_+^2+\Lambda_-^2\right)}}+k\right]>3l, \\ &r_2=-l\left[2\sqrt{\frac{k^2+3}{4}+\frac{3v_0^2}{l^2\left(\Lambda_+^2+\Lambda_-^2\right)}}-k\right]<0,\qquad k=\frac{\Lambda_+^2-\Lambda_-^2}{\Lambda_+^2+\Lambda_-^2}. \end{aligned} \tag{4.6}$$

In accordance with (3.16), we obtain the following expression for the string solution ($\Delta r = r-3l$, $\Delta r_1 = r_1 - 3l$)

$$\begin{aligned} \xi^1(r) = &\frac{8}{\left(\Lambda_+^2+\Lambda_-^2\right)^{1/2}}\left[\frac{l\Delta r}{(3l-r_2)\Delta r_1}\right]^{1/2}\times \\ &F_D^{(5)}\left(1/2;-1/2,-1/2,1/2,1/2,1/2;3/2;-\frac{\Delta r}{2l},-\frac{\Delta r}{4l},-\frac{\Delta r}{6l},-\frac{\Delta r}{3l-r_2},\frac{\Delta r}{\Delta r_1}\right), \end{aligned} \tag{4.7}$$

where $F_D^{(5)}$ is hypergeometric function of five variables.[4]

[3] For all string and D2-brane solutions we are considering here, $r_{min} = 9r_0/2 \equiv 3l$.

[4] The definition and some properties of the hypergeometric functions $F_D^{(n)}(a;b_1,\ldots,b_n;c;z_1,\ldots,z_n)$ are given in the appendix.

Now, we can compute the conserved momenta on the obtained solution. According to (3.17), they are ($E = -P_0$):

$$\frac{E}{\Lambda_0^0} = \frac{P_I}{\Lambda_0^I} = T\left[\frac{2^7 l\Delta r_1}{\left(\Lambda_+^2 + \Lambda_-^2\right)(3l - r_2)}\right]^{1/2}\left(1 + \frac{\Delta r_1}{3l - r_2}\right)^{-1/2} \tag{4.8}$$
$$\times F_D^{(1)}\left(1/2; 1/2; 3/2; \frac{1}{1 + \frac{3l - r_2}{\Delta r_1}}\right),$$

$$P_{\theta_1} = \left(\Lambda_0^{\theta_1} - \Lambda_0^{\theta_2}\cos\psi_1^0\right) I_A + \left(\Lambda_0^{\theta_1} + \Lambda_0^{\theta_2}\cos\psi_1^0\right) I_B, \tag{4.9}$$
$$P_{\theta_2} = \left(\Lambda_0^{\theta_2} - \Lambda_0^{\theta_1}\cos\psi_1^0\right) I_A + \left(\Lambda_0^{\theta_2} + \Lambda_0^{\theta_1}\cos\psi_1^0\right) I_B,$$

where

$$I_A = T\left[\frac{2^7 l^5 \Delta r_1}{\left(\Lambda_+^2 + \Lambda_-^2\right)(3l - r_2)}\right]^{1/2}\left(1 + \frac{\Delta r_1}{2l}\right)\left(1 + \frac{\Delta r_1}{6l}\right)\left(1 + \frac{\Delta r_1}{3l - r_2}\right)^{-1/2} \tag{4.10}$$
$$\times F_D^{(3)}\left(1/2; -1, -1, 1/2; 3/2; \frac{1}{1 + \frac{2l}{\Delta r_1}}, \frac{1}{1 + \frac{6l}{\Delta r_1}}, \frac{1}{1 + \frac{3l - r_2}{\Delta r_1}}\right),$$

$$I_B = \frac{T}{9}\left[\frac{2^9 \left(l\Delta r_1\right)^3}{\left(\Lambda_+^2 + \Lambda_-^2\right)(3l - r_2)}\right]^{1/2}\left(1 + \frac{\Delta r_1}{4l}\right)\left(1 + \frac{\Delta r_1}{3l - r_2}\right)^{-1/2} \tag{4.11}$$
$$\times F_D^{(2)}\left(1/2; -1, 1/2; 5/2; \frac{1}{1 + \frac{4l}{\Delta r_1}}, \frac{1}{1 + \frac{3l - r_2}{\Delta r_1}}\right).$$

Our next task is to find the relation between the energy E and the other conserved quantities P_I, P_{θ_1}, P_{θ_2}, in the semiclassical limit (large conserved charges), which corresponds to $r_1 \to \infty$. In this limit,

$$\frac{E}{\Lambda_0^0} = \frac{P_I}{\Lambda_0^I} = \frac{\pi T \left(2^3 l\right)^{1/2}}{\left(\Lambda_+^2 + \Lambda_-^2\right)^{1/2}}, \qquad I_A = I_B = \frac{\pi T \left(2l\right)^{1/2} v_0^2}{\left(\Lambda_+^2 + \Lambda_-^2\right)^{3/2}},$$

which leads to

$$E^2 = \mathbf{P}^2 + 2\pi T \left(6r_0\right)^{1/2}\left(P_{\theta_1}^2 + P_{\theta_2}^2\right)^{1/2}, \qquad \mathbf{P}^2 = \delta_{IJ} P_I P_J. \tag{4.12}$$

This is a generalization of the energy-charge relation $E \sim K^{1/2}$ for the case $P_I \neq 0$ and two conserved angular momenta P_{θ_1}, P_{θ_2}. Thus, the above string configuration has the same semiclassical behavior as the membrane in (4.20) of [11], which is given by the relation

$$E^2 = \mathbf{P}^2 + 2\sqrt{6}\pi^2 T_{M2} l_{11}^3 \mid \mathbf{\Lambda}_1 \mid \left(P_\theta^2 + P_{\tilde{\theta}}^2\right)^{1/2}.$$

Now, let us consider rotating string on the background (4.2). To ensure that the conditions $\mathcal{P}_\mu = 0$ and the second constraint in (3.15) are satisfied, we have to choose the following embedding

$$X^0 = \Lambda_0^0 \xi^0, \quad X^I = \Lambda_0^I \xi^0, \quad r = r(\xi^1), \quad \theta_1 = \Lambda_0^\theta \xi^0, \quad \phi_2 = \Lambda_0^\phi \xi^0. \tag{4.13}$$

This ansatz is analogous to the previous one, with θ_2 replaced by ϕ_2. The first constraint in (3.15) now reads,

$$g_{rr}r'^2 - \mathcal{U} = \frac{r_0^{1/2}}{C}r'^2 - r_0^{1/2}C\left(v_0^2 - \bar\Lambda_-^2 A^2 - \bar\Lambda_+^2 B^2 - \Lambda_D^2 D^2\right) = 0, \tag{4.14}$$

where v_0^2 is given in (4.5) and

$$\begin{aligned}\bar\Lambda_\pm^2 &= \left(\Lambda_0^\theta\right)^2 + \left(\Lambda_0^\phi\right)^2 \sin^2\theta_2^0 \pm 2\Lambda_0^\theta\Lambda_0^\phi \sin\psi_1^0 \sin\theta_2^0,\\ \Lambda_D^2 &= \left(\Lambda_0^\phi\right)^2 \cos^2\theta_2^0.\end{aligned} \tag{4.15}$$

From here, one obtains the solutions of the equation $r' = 0$:

$$r_{min} = 9r_0/2 \equiv 3l, \qquad r_{max} = r_1 > 3l, \qquad r_2 < 0.$$

The rotating string solution $\xi^1(r)$ expresses through the same hypergeometric function as in (4.7), but now depends on different parameters

$$\begin{aligned}\xi^1(r) &= \frac{8}{\left(\bar\Lambda_+^2 + \bar\Lambda_-^2 + 4\Lambda_D^2/3\right)^{1/2}}\left[\frac{l\Delta r}{(3l-r_2)\Delta r_1}\right]^{1/2} \times \\ &F_D^{(5)}\left(1/2; -1/2, -1/2, 1/2, 1/2, 1/2; 3/2; -\frac{\Delta r}{2l}, -\frac{\Delta r}{4l}, -\frac{\Delta r}{6l}, -\frac{\Delta r}{3l-r_2}, \frac{\Delta r}{\Delta r_1}\right).\end{aligned} \tag{4.16}$$

The same is true for E and P_I (compare with (4.8))

$$\begin{aligned}\frac{E}{\Lambda_0^0} = \frac{P_I}{\Lambda_0^I} &= 8T\left[\frac{2l\Delta r_1}{\left(\bar\Lambda_+^2 + \bar\Lambda_-^2 + 4\Lambda_D^2/3\right)(3l-r_2)}\right]^{1/2}\left(1 + \frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\\ &\times F_D^{(1)}\left(1/2; 1/2; 3/2; \frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right).\end{aligned} \tag{4.17}$$

For the conserved angular momenta P_θ and P_ϕ, (3.17) gives

$$\begin{aligned}P_\theta &= \left(\Lambda_0^\theta - \Lambda_0^\phi \sin\psi_1^0 \sin\theta_2^0\right)J_A + \left(\Lambda_0^\theta + \Lambda_0^\phi \sin\psi_1^0 \sin\theta_2^0\right)J_B, \\ P_\phi &= \left(\Lambda_0^\phi \sin\theta_2^0 - \Lambda_0^\theta \sin\psi_1^0\right)\sin\theta_2^0 J_A + \left(\Lambda_0^\phi \sin\theta_2^0 + \Lambda_0^\theta \sin\psi_1^0\right)\sin\theta_2^0 J_B\\ &\quad + \Lambda_0^\phi \cos^2\theta_2^0 J_D,\end{aligned} \tag{4.18}$$

where

$$\begin{aligned}J_A &= 8T\left[\frac{2l^5\Delta r_1}{\left(\bar\Lambda_+^2 + \bar\Lambda_-^2 + 4\Lambda_D^2/3\right)(3l-r_2)}\right]^{1/2}\left(1+\frac{\Delta r_1}{2l}\right)\left(1+\frac{\Delta r_1}{6l}\right)\\ &\times\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2} F_D^{(3)}\left(1/2; -1, -1, 1/2; 3/2; \frac{1}{1+\frac{2l}{\Delta r_1}}, \frac{1}{1+\frac{6l}{\Delta r_1}}, \frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right),\end{aligned} \tag{4.19}$$

$$J_B = \frac{16}{9}T\left[\frac{2(l\Delta r_1)^3}{\left(\bar{\Lambda}_+^2+\bar{\Lambda}_-^2+4\Lambda_D^2/3\right)(3l-r_2)}\right]^{1/2}\left(1+\frac{\Delta r_1}{4l}\right)\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}$$
$$\times F_D^{(2)}\left(1/2;-1,1/2;5/2;\frac{1}{1+\frac{4l}{\Delta r_1}},\frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right), \quad (4.20)$$

$$J_D = 8T\left[\frac{2l^5\Delta r_1}{\left(\bar{\Lambda}_+^2+\bar{\Lambda}_-^2+4\Lambda_D^2/3\right)(3l-r_2)}\right]^{1/2}\left(1+\frac{\Delta r_1}{3l}\right)\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}$$
$$\times F_D^{(2)}\left(1/2;-2,1/2;3/2;\frac{1}{1+\frac{3l}{\Delta r_1}},\frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right). \quad (4.21)$$

In the semiclassical limit $r_1 \to \infty$, one gets the following dependence of the energy on the charges P_I, P_θ and P_ϕ

$$E^2 = \mathbf{P}^2 + 2\pi T\,(6r_0)^{1/2}\left(P_\theta^2 + \frac{3P_\phi^2}{3-\cos^2\theta_2^0}\right)^{1/2}. \quad (4.22)$$

Again, this is a generalization of the energy-charge relation $E \sim K^{1/2}$, and for $\theta_2^0 = \pi/2$ (4.22) has the same form as the expression in (4.12).

Now, we turn to the case of string rotating in the background given in (4.3). To satisfy the conditions $\mathcal{P}_\mu = 0$ and the second constraint in (3.15), we use the ansatz

$$X^0 = \Lambda_0^0\xi^0, \quad X^I = \Lambda_0^I\xi^0, \quad r = r(\xi^1), \quad \phi_1 = \Lambda_0^{\phi_1}\xi^0, \quad \phi_2 = \Lambda_0^{\phi_2}\xi^0. \quad (4.23)$$

This embedding is analogous to the one just considered, where θ_1 is replaced by ϕ_1.

The first constraint in (3.15) takes the form,

$$g_{rr}r'^2 - \mathcal{U} = \frac{r_0^{1/2}}{C}r'^2 - r_0^{1/2}C\left(v_0^2 - \tilde{\Lambda}_+^2A^2 - \tilde{\Lambda}_-^2B^2 - \tilde{\Lambda}_D^2D^2\right) = 0, \quad (4.24)$$

where v_0^2 is the same as before, and

$$\tilde{\Lambda}_\pm^2 = \left(\Lambda_0^{\phi_1}\right)^2\sin^2\theta_1^0 + \left(\Lambda_0^{\phi_2}\right)^2\sin^2\theta_2^0 \pm 2\Lambda_0^{\phi_1}\Lambda_0^{\phi_2}\cos\psi_1^0\sin\theta_1^0\sin\theta_2^0,$$
$$\tilde{\Lambda}_D^2 = \left(\Lambda_0^{\phi_1}\cos\theta_1^0 + \Lambda_0^{\phi_2}\cos\theta_2^0\right)^2. \quad (4.25)$$

Since (4.24) can be obtained from (4.14) by the replacements $\bar{\Lambda}_\pm^2 \to \tilde{\Lambda}_\mp^2$, $\Lambda_D^2 \to \tilde{\Lambda}_D^2$, in the same way one can receive the new values for $r_{max} = r_1$ and r_2, the new string solution from (4.16), the new expressions for the energy E and the momenta P_I from (4.17). In accordance with (3.17), the conserved angular momenta P_{ϕ_1} and P_{ϕ_2} are given by

$$P_{\phi_1} = \left(\Lambda_0^{\phi_1}\sin\theta_1^0 + \Lambda_0^{\phi_2}\cos\psi_1^0\sin\theta_2^0\right)\sin\theta_1^0K_A \quad (4.26)$$
$$+ \left(\Lambda_0^{\phi_1}\sin\theta_1^0 - \Lambda_0^{\phi_2}\cos\psi_1^0\sin\theta_2^0\right)\sin\theta_1^0K_B$$
$$+ \left(\Lambda_0^{\phi_1}\cos\theta_1^0 + \Lambda_0^{\phi_2}\cos\theta_2^0\right)\cos\theta_1^0K_D,$$

$$
\begin{aligned}
P_{\phi_2} =& \left(\Lambda_0^{\phi_2}\sin\theta_2^0 + \Lambda_0^{\phi_1}\cos\psi_1^0\sin\theta_1^0\right)\sin\theta_2^0 K_A \\
&+ \left(\Lambda_0^{\phi_2}\sin\theta_2^0 - \Lambda_0^{\phi_1}\cos\psi_1^0\sin\theta_1^0\right)\sin\theta_2^0 K_B \\
&+ \left(\Lambda_0^{\phi_1}\cos\theta_1^0 + \Lambda_0^{\phi_2}\cos\theta_2^0\right)\cos\theta_2^0 K_D,
\end{aligned}
\tag{4.27}
$$

where K_A, K_B, K_D can be obtained from (4.19), (4.20), (4.21), by the above mentioned replacements.

Taking the semiclassical limit ($r_1 \to \infty$) in the expressions for E, P_I, P_{ϕ_1} and P_{ϕ_2}, after some calculations, one receives the following relation between them

$$
\begin{aligned}
E^2 =& \mathbf{P}^2 + 2\pi T\left(\frac{6r_0}{\Delta}\right)^{1/2} \times \\
&\left[\left(3-\cos^2\theta_2^0\right)P_{\phi_1}^2 + \left(3-\cos^2\theta_1^0\right)P_{\phi_2}^2 - 4P_{\phi_1}P_{\phi_2}\cos\theta_1^0\cos\theta_2^0\right]^{1/2},
\end{aligned}
\tag{4.28}
$$

where

$$
\Delta = 3 - \cos^2\theta_1^0 - \cos^2\theta_2^0 - \cos^2\theta_1^0\cos^2\theta_2^0.
$$

This is another generalization of the energy-charge relation $E \sim K^{1/2}$, and for $\theta_1^0 = \theta_2^0 = \pi/2$ has the same form as the relation in (4.12).

The equality (4.28) is only valid for $\Delta \neq 0$. To see what will be the semiclassical behavior of the rotating string configuration for $\Delta = 0$, let us consider the particular case $\theta_1^0 = \theta_2^0 = 0$. According to (4.26), (4.27), the two angular momenta become equal, $P_{\phi_1} = P_{\phi_2} \equiv P_\phi$. Performing the necessary computations, one arrives at

$$
E^2 = \mathbf{P}^2 + 6\pi T r_0^{1/2} P_\phi, \tag{4.29}
$$

which describes the same type of semiclassical behavior.

Comparing (4.4), (4.13) and (4.23) with each other, one sees that none of them represents string configuration with nontrivial wrapping. Then a natural question is if such solutions do exist at all. The analysis shows that the reason for the absence of wrapping is that we have too many restrictions on the embedding parameters Λ_m^μ for the backgrounds (4.1), (4.2) and (4.3). However, it turns out that if we restrict ourselves to particular cases of these backgrounds by fixing the values of part of the angles $\theta_{1,2}^0$, ψ_1^0 or $\phi_{1,2}^0$, we can obtain wrapped rotating string solutions. An example of such solution is given by the ansatz

$$
\begin{aligned}
&X^0 = \Lambda_0^0\xi^0, \quad X^I = \Lambda_0^I\xi^0, \quad r = r(\xi^1), \quad \theta_1^0 = \theta_2^0 = 0, \\
&\psi_1 = \Lambda_0^{\psi_1}\xi^0 - (\Lambda_1^{\phi_1} + \Lambda_1^{\phi_2})\xi^1, \quad \phi_1 = \Lambda_0^{\phi_1}\xi^0 + \Lambda_1^{\phi_1}\xi^1, \quad \phi_2 = \Lambda_0^{\phi_2}\xi^0 + \Lambda_1^{\phi_2}\xi^1.
\end{aligned}
$$

The background metric felt by the string is

$$
ds^2 = r_0^{1/2}\left\{C\left[-(dx^0)^2 + \delta_{IJ}dx^I dx^J + D^2 d(\psi_1+\phi_1+\phi_2)^2\right] + \frac{dr^2}{C}\right\}.
$$

It can be seen as particular case of (4.3) after the replacement $(\psi_1+\phi_1+\phi_2) \to (\phi_1+\phi_2)$. The calculations lead to the same result about the semiclassical behavior of this wrapped string configuration as in (4.29), where P_ϕ must be replaced with P_{ψ_1}, P_{ϕ_1}, or P_{ϕ_2}, which are equal to each other.

Another example of wrapped string solution is

$$X^0 = \Lambda_0^0 \xi^0, \quad X^I = \Lambda_0^I \xi^0, \quad r = r(\xi^1), \quad \phi_1^0 = \theta_2^0 = 0,$$
$$\theta_1 = \Lambda_0^{\theta_1} \xi^0, \quad \psi_1 = \Lambda_0^{\psi_1} \xi^0 + \Lambda_1^{\psi_1} \xi^1, \quad \phi_2 = \Lambda_0^{\phi_2} \xi^0 - \Lambda_1^{\psi_1} \xi^1.$$

The background seen by the string now is

$$ds^2 = r_0^{1/2} \left\{ C \left[-(dx^0)^2 + \delta_{IJ} dx^I dx^J + (A^2 + B^2) d\theta_1^2 + D^2 d(\psi_1 + \phi_2)^2 \right] + \frac{dr^2}{C} \right\},$$

which can be considered as particular case of (4.2) after the replacement $(\psi_1 + \phi_2) \to \phi_2$. In the semiclassical limit, for the above string configuration, one receives the following energy-charge relation ($P_{\psi_1} = P_{\phi_2}$)

$$E^2 = \mathbf{P}^2 + 2\pi T \left(3r_0\right)^{1/2} \left(2P_{\theta_1}^2 + 3P_{\psi_1}^2\right)^{1/2}. \tag{4.30}$$

It is particular case of (4.22).

Let us finally note that in considering the semiclassical limit (large charges), we take into account only the leading terms in the expressions for the conserved quantities. However, there is no problem to include the higher order terms. For instance, the inclusion of the next-to-leading order term, modifies (4.30) to

$$E^2 = \mathbf{P}^2 + 2\pi T \left(3r_0\right)^{1/2} \left(2P_{\theta_1}^2 + 3P_{\psi_1}^2\right)^{1/2} - \frac{1}{2}(\pi T)^2 (3r_0)^3 \frac{P_{\theta_1}^2}{2P_{\theta_1}^2 + 3P_{\psi_1}^2}. \tag{4.31}$$

5. Rotating D2-brane solutions, conserved charges and their semiclassical limits

In this section, we will consider D2-branes rotating in the backgrounds (4.1), (4.2) and (4.3), as it was already done for strings. It turns out that for every one of these three backgrounds, there exist two D2-brane configurations of the type (3.1), which ensure that the equalities (3.34), (3.35) and $\mathcal{P}_\mu^2 = 0$ are identically satisfied.

We begin with the following D2-brane embedding in the target space metric (4.1):

$$X^0 = \Lambda_0^0 \xi^0 + \frac{(\mathbf{\Lambda}_0.\mathbf{\Lambda}_1)}{\Lambda_0^0} \left(\xi^1 + c\xi^2\right), \quad X^I = \Lambda_0^I \xi^0 + \Lambda_1^I \left(\xi^1 + c\xi^2\right), \tag{5.1}$$
$$r = r(\xi^2), \quad \theta_1 = \Lambda_0^{\theta_1} \xi^0, \quad \theta_2 = \Lambda_0^{\theta_2} \xi^0; \quad (\mathbf{\Lambda}_0.\mathbf{\Lambda}_1) = \delta_{IJ} \Lambda_0^I \Lambda_1^J, \quad c = constant.$$

It corresponds to D2-brane extended in the radial direction r, and rotating in the planes given by the angles θ_1 and θ_2 with constant angular momenta P_{θ_1} and P_{θ_2}. It is nontrivially spanned along x^0 and x^I and moves with constant energy E, and constant momenta P_I.

The metric induced on the D2-brane worldvolume is

$$G_{00} = -r_0^{1/2} C \left(v_0^2 - \Lambda_-^2 A^2 - \Lambda_+^2 B^2\right),$$
$$G_{11} = r_0^{1/2} MC, \quad G_{12} = cG_{11}, \quad G_{22} = g_{rr} r'^2 + c^2 G_{11},$$

where v_0^2 and $\Lambda_\pm$ are defined in (4.5) and

$$M = \mathbf{\Lambda}_1^2 - \frac{(\mathbf{\Lambda}_0.\mathbf{\Lambda}_1)^2}{\left(\Lambda_0^0\right)^2}. \tag{5.2}$$

The Lagrangian (3.27) takes the form

$$\mathcal{L}^A(\xi^2) = \frac{1}{4\lambda^0}\left(\tilde{K}_{rr}r'^2 - \tilde{V}\right), \qquad \tilde{K}_{rr} = -(2\lambda^0 T_{D2})^2 r_0 M e^{-\Phi},$$
$$\tilde{V} = r_0^{1/2} C\left(v_0^2 - \Lambda_-^2 A^2 - \Lambda_+^2 B^2\right) e^{-\Phi}.$$

From the yet unsolved constraint (3.33)

$$\tilde{K}_{rr}r'^2 + \tilde{V} = 0,$$

one obtains the turning points of the effective one-dimensional periodic motion by solving the equation $r' = 0$. In the case under consideration, the result is given in (4.6).

Applying the general formula (3.36), we obtain the following expression for the D2-brane solution

$$\xi^2(r) = \int_{3l}^{r}\left[-\frac{\tilde{K}_{rr}(t)}{\tilde{V}(t)}\right]^{1/2} dt = \frac{16}{3}\lambda^0 T_{D2}\left[\frac{Ml}{\left(\Lambda_+^2 + \Lambda_-^2\right)(3l - r_2)\,\Delta r_1}\right]^{1/2} (2\Delta r)^{3/4} \times$$
$$F_D^{(5)}\left(3/4; -1/4, -1/4, 1/4, 1/2, 1/2; 7/4; -\frac{\Delta r}{2l}, -\frac{\Delta r}{4l}, -\frac{\Delta r}{6l}, -\frac{\Delta r}{3l - r_2}, \frac{\Delta r}{\Delta r_1}\right). \tag{5.3}$$

Now, we compute the conserved momenta on the obtained solution according to (3.37):

$$\frac{E}{\Lambda_0^0} = \frac{P_I}{\Lambda_0^I} = 8\pi^2 T_{D2}\left[\frac{Ml}{\left(\Lambda_+^2 + \Lambda_-^2\right)(3l - r_2)}\right]^{1/2} \times \tag{5.4}$$
$$\left(1 + \frac{\Delta r_1}{2l}\right)^{1/2}\left(1 + \frac{\Delta r_1}{4l}\right)^{1/2}\left(1 + \frac{\Delta r_1}{6l}\right)^{-1/2}\left(1 + \frac{\Delta r_1}{3l - r_2}\right)^{-1/2} \times$$
$$F_D^{(4)}\left(1/2; -1/2, -1/2, 1/2, 1/2; 1; \frac{1}{1 + \frac{2l}{\Delta r_1}}, \frac{1}{1 + \frac{4l}{\Delta r_1}}, \frac{1}{1 + \frac{6l}{\Delta r_1}}, \frac{1}{1 + \frac{3l - r_2}{\Delta r_1}}\right),$$

$$P_{\theta_1} = \left(\Lambda_0^{\theta_1} - \Lambda_0^{\theta_2}\cos\psi_1^0\right) I_{A1}^D + \left(\Lambda_0^{\theta_1} + \Lambda_0^{\theta_2}\cos\psi_1^0\right) I_{B1}^D, \tag{5.5}$$
$$P_{\theta_2} = \left(\Lambda_0^{\theta_2} - \Lambda_0^{\theta_1}\cos\psi_1^0\right) I_{A1}^D + \left(\Lambda_0^{\theta_2} + \Lambda_0^{\theta_1}\cos\psi_1^0\right) I_{B1}^D,$$

where

$$I_{A1}^D = 8\pi^2 T_{D2}\left[\frac{Ml^5}{\left(\Lambda_+^2 + \Lambda_-^2\right)(3l - r_2)}\right]^{1/2} \times \tag{5.6}$$
$$\left(1 + \frac{\Delta r_1}{2l}\right)^{3/2}\left(1 + \frac{\Delta r_1}{4l}\right)^{1/2}\left(1 + \frac{\Delta r_1}{6l}\right)^{1/2}\left(1 + \frac{\Delta r_1}{3l - r_2}\right)^{-1/2} \times$$
$$F_D^{(4)}\left(1/2; -3/2, -1/2, -1/2, 1/2; 1; \frac{1}{1 + \frac{2l}{\Delta r_1}}, \frac{1}{1 + \frac{4l}{\Delta r_1}}, \frac{1}{1 + \frac{6l}{\Delta r_1}}, \frac{1}{1 + \frac{3l - r_2}{\Delta r_1}}\right),$$

$$I_{B1}^{D} = \frac{4}{3}\pi^2 T_{D2}\left[\frac{Ml^3}{\left(\Lambda_+^2+\Lambda_-^2\right)(3l-r_2)}\right]^{1/2}\times \tag{5.7}$$
$$\Delta r_1\left(1+\frac{\Delta r_1}{2l}\right)^{1/2}\left(1+\frac{\Delta r_1}{4l}\right)^{3/2}\left(1+\frac{\Delta r_1}{6l}\right)^{-1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\times$$
$$F_D^{(4)}\left(1/2;-1/2,-3/2,1/2,1/2;2;\frac{1}{1+\frac{2l}{\Delta r_1}},\frac{1}{1+\frac{4l}{\Delta r_1}},\frac{1}{1+\frac{6l}{\Delta r_1}},\frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right).$$

In the semiclassical limit, (5.4) - (5.7) simplify to

$$\frac{E}{\Lambda_0^0}=\frac{P_I}{\Lambda_0^I}=\frac{2}{3}\pi^2 T_{D2}\left(\frac{M}{\Lambda_+^2+\Lambda_-^2}\right)^{1/2},$$
$$P_{\theta_1}=2\Lambda_0^{\theta_1}I_{A1}^D,\quad P_{\theta_2}=2\Lambda_0^{\theta_2}I_{A1}^D,\quad I_{A1}^D=I_{B1}^D=\frac{\sqrt{3}\pi^2 T_{D2}M^{1/2}}{\left(\Lambda_+^2+\Lambda_-^2\right)^{3/2}}v_0^2.$$

From here, one obtains the following relation between the energy and the conserved charges

$$E^2\left(E^2-\mathbf{P}^2\right)^2-\frac{2^3}{3^5}(\pi^2 T_{D2})^2\left[\mathbf{\Lambda}_1^2E^2-(\mathbf{\Lambda}_1.\mathbf{P})^2\right]\left(P_{\theta_1}^2+P_{\theta_2}^2\right)=0, \tag{5.8}$$

which is third order algebraic equation for E^2. Therefore, this D2-brane configuration reproduces particular case of the M2-brane semiclassical behavior given in (4.19) of [11]

$$\left\{E^2\left(E^2-\mathbf{P}^2\right)-(2\pi^2T_{M2}l_{11}^3)^2\left\{(\mathbf{\Lambda}_1\times\mathbf{\Lambda}_2)^2E^2-[(\mathbf{\Lambda}_1\times\mathbf{\Lambda}_2)\times\mathbf{P}]^2\right\}\right\}^2$$
$$-6(2\pi^2T_{M2}l_{11}^3)^2E^2\left[\mathbf{\Lambda}_1^2E^2-(\mathbf{\Lambda}_1.\mathbf{P})^2\right]\left(P_\theta^2+P_{\tilde{\theta}}^2\right)=0,$$

corresponding to $(\mathbf{\Lambda}_1\times\mathbf{\Lambda}_2)=0$. For $(\mathbf{\Lambda}_1.\mathbf{P})=0$, (5.8) reduces to

$$E^2=\mathbf{P}^2+\frac{2^{3/2}}{3^{5/2}}\pi^2T_{D2}\mid\mathbf{\Lambda}_1\mid\left(P_{\theta_1}^2+P_{\theta_2}^2\right)^{1/2}.$$

This is the same type energy-charge relation as the one obtained for the string in (4.12).

Let us now consider the other possible D2-brane embedding for the same background metric (4.1). It is given by

$$X^0=\Lambda_0^0\xi^0,\quad X^I=\Lambda_0^I\xi^0,\quad r=r(\xi^2), \tag{5.9}$$
$$\theta_1=\Lambda_0^\theta\xi^0+\Lambda_1^\theta\xi^1+\Lambda_2^\theta\xi^2,\quad \theta_2=\Lambda_0^\theta\xi^0-\Lambda_1^\theta\xi^1-\Lambda_2^\theta\xi^2.$$

This ansatz describes D2-brane, which is extended along the radial direction r and rotates in the planes defined by the angles θ_1 and θ_2, with equal angular momenta $P_{\theta_1}=P_{\theta_2}=P_\theta$. Now we have nontrivial wrapping along θ_1 and θ_2. In addition, the D2-brane moves along x^0 and x^I with constant energy E and constant momenta P_I respectively.

For the present case, the Lagrangian (3.27) reduces to

$$\mathcal{L}^A(\xi^2)=\frac{1}{4\lambda^0}\left(\tilde{K}_{rr}r'^2-\tilde{V}\right),\quad \tilde{K}_{rr}=-(2\lambda^0T_{D2})^2r_0\left(\Lambda_{1+}^2A^2+\Lambda_{1-}^2B^2\right)e^{-\Phi},$$
$$\tilde{V}=r_0^{1/2}C\left(v_0^2-\check{\Lambda}_-^2A^2-\check{\Lambda}_+^2B^2\right)e^{-\Phi},$$

where

$$\Lambda_{1\pm}^2 = 2\left(\Lambda_1^\theta\right)^2\left(1 \pm \cos\psi_1^0\right), \qquad \check{\Lambda}_{\pm}^2 = 2\left(\Lambda_0^\theta\right)^2\left(1 \pm \cos\psi_1^0\right).$$

The constraint (3.33)

$$\tilde{K}_{rr} r'^2 + \tilde{V} = 0,$$

leads to the same solutions of the equation $r' = 0$, as given in (4.6), but in terms of the new parameters $\check{\Lambda}_\pm$ instead of $\Lambda_\pm$.

Replacing the above expressions for $\tilde{K}_{rr}$ and $\tilde{V}$ in (3.36), we obtain the D2-brane solution:

$$\begin{aligned}
&\xi^2(r) = \frac{8}{3}\lambda^0 T_{D2}\left[\frac{l\left(\Lambda_{1+}^2 + \Lambda_{1-}^2\right)(3l - v_+)(3l - v_-)}{3\left(\check{\Lambda}_+^2 + \check{\Lambda}_-^2\right)(3l - r_2)\,\Delta r_1}\right]^{1/2}(2\Delta r)^{3/4}\times \\
&F_D^{(7)}\left(3/4; -1/4, -1/4, 1/4, -1/2, -1/2, 1/2, 1/2; 7/4;\right. \\
&\left.-\frac{\Delta r}{2l}, -\frac{\Delta r}{4l}, -\frac{\Delta r}{6l}, -\frac{\Delta r}{3l - v_+}, -\frac{\Delta r}{3l - v_-}, -\frac{\Delta r}{3l - r_2}, \frac{\Delta r}{\Delta r_1}\right),
\end{aligned} \tag{5.10}$$

where $v_\pm$ are the zeros of the polynomial

$$t^2 - 2l\frac{\Lambda_{1+}^2 - \Lambda_{1-}^2}{\Lambda_{1+}^2 + \Lambda_{1-}^2}t - 3l^2 = (t - v_+)(t - v_-).$$

In the case under consideration, the conserved quantities are E, P_I and P_θ. By using (3.37), we derive the following result for them

$$\begin{aligned}
&\frac{E}{\Lambda_0^0} = \frac{P_I}{\Lambda_0^I} = 4\pi^2 T_{D2}\left[\frac{l\left(\Lambda_{1+}^2 + \Lambda_{1-}^2\right)(3l - v_+)(3l - v_-)}{3\left(\check{\Lambda}_+^2 + \check{\Lambda}_-^2\right)(3l - r_2)}\right]^{1/2}\times \\
&\left(1 + \frac{\Delta r_1}{2l}\right)^{1/2}\left(1 + \frac{\Delta r_1}{4l}\right)^{1/2}\left(1 + \frac{\Delta r_1}{6l}\right)^{-1/2}\times \\
&\left(1 + \frac{\Delta r_1}{3l - v_+}\right)^{1/2}\left(1 + \frac{\Delta r_1}{3l - v_-}\right)^{1/2}\left(1 + \frac{\Delta r_1}{3l - r_2}\right)^{-1/2}\times \\
&F_D^{(6)}\left(1/2; -1/2, -1/2, 1/2, -1/2, -1/2, 1/2; 1;\right. \\
&\left.\frac{1}{1 + \frac{2l}{\Delta r_1}}, \frac{1}{1 + \frac{4l}{\Delta r_1}}, \frac{1}{1 + \frac{6l}{\Delta r_1}}, \frac{1}{1 + \frac{3l - v+}{\Delta r_1}}, \frac{1}{1 + \frac{3l - v-}{\Delta r_1}} \frac{1}{1 + \frac{3l - r_2}{\Delta r_1}}\right),
\end{aligned} \tag{5.11}$$

$$P_\theta = \Lambda_0^\theta\left[\left(1 - \cos\psi_1^0\right)I_{A2}^D + \left(1 + \cos\psi_1^0\right)I_{B2}^D\right],$$

where

$$I_{A2}^D = 4\pi^2 T_{D2}\left[\frac{l^5\left(\Lambda_{1+}^2 + \Lambda_{1-}^2\right)(3l - v_+)(3l - v_-)}{3\left(\check{\Lambda}_+^2 + \check{\Lambda}_-^2\right)(3l - r_2)}\right]^{1/2}\times$$

$$\left(1+\frac{\Delta r_1}{2l}\right)^{3/2}\left(1+\frac{\Delta r_1}{4l}\right)^{1/2}\left(1+\frac{\Delta r_1}{6l}\right)^{1/2}\times$$
$$\left(1+\frac{\Delta r_1}{3l-v_+}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-v_-}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\times$$
$$F_D^{(6)}\,(1/2;-3/2,-1/2,-1/2,-1/2,-1/2,1/2;1;$$
$$\left.\frac{1}{1+\frac{2l}{\Delta r_1}},\frac{1}{1+\frac{4l}{\Delta r_1}},\frac{1}{1+\frac{6l}{\Delta r_1}},\frac{1}{1+\frac{3l-v+}{\Delta r_1}},\frac{1}{1+\frac{3l-v-}{\Delta r_1}}\frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right),$$

$$I_{B2}^D=2\pi^2T_{D2}\left[\frac{l^3\left(\Lambda_{1+}^2+\Lambda_{1-}^2\right)(3l-v_+)(3l-v_-)}{3^3\left(\check{\Lambda}_+^2+\check{\Lambda}_-^2\right)(3l-r_2)}\right]^{1/2}\times$$
$$\Delta r_1\left(1+\frac{\Delta r_1}{2l}\right)^{1/2}\left(1+\frac{\Delta r_1}{4l}\right)^{3/2}\left(1+\frac{\Delta r_1}{6l}\right)^{-1/2}\times$$
$$\left(1+\frac{\Delta r_1}{3l-v_+}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-v_-}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\times$$
$$F_D^{(6)}\,(1/2;-1/2,-3/2,1/2,-1/2,-1/2,1/2;2;$$
$$\left.\frac{1}{1+\frac{2l}{\Delta r_1}},\frac{1}{1+\frac{4l}{\Delta r_1}},\frac{1}{1+\frac{6l}{\Delta r_1}},\frac{1}{1+\frac{3l-v+}{\Delta r_1}},\frac{1}{1+\frac{3l-v-}{\Delta r_1}}\frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right).$$

Taking the semiclassical limit in the above expressions[5], we obtain the following dependence of the energy on P_I and P_θ:

$$E^2=\mathbf{P}^2+3^{5/3}(2\pi T_{D2}\Lambda_1^\theta)^{2/3}P_\theta^{4/3}. \tag{5.12}$$

This is the same semiclassical behavior as the one exhibited by the M2-brane as given in (4.27) of [11]:

$$E^2=\mathbf{P}^2+3^{5/3}(2\pi T_{M2}l_{11}^3\Lambda_1^6)^{2/3}P_\theta^{4/3},$$

which is a generalization of the energy-charge relation $E\sim K^{2/3}$ for the case $\mathbf{P}\neq 0$.

Now, we turn to the case of D2-branes rotating in the background (4.2). Again, we have two possible embeddings of the type (3.1). The first one is given by the ansatz

$$X^0=\Lambda_0^0\xi^0+\frac{(\mathbf{\Lambda}_0.\mathbf{\Lambda}_1)}{\Lambda_0^0}\left(\xi^1+c\xi^2\right),\qquad X^I=\Lambda_0^I\xi^0+\Lambda_1^I\left(\xi^1+c\xi^2\right), \tag{5.13}$$
$$r=r(\xi^2),\qquad \theta_1=\Lambda_0^\theta\xi^0,\qquad \phi_2=\Lambda_0^\phi\xi^0.$$

(5.13) is analogous to (5.1), but now the rotations are in the planes defined by the angles θ_1 and ϕ_2 instead of θ_1 and θ_2.

The Lagrangian (3.27) takes the form

$$\mathcal{L}^A(\xi^2)=\frac{1}{4\lambda^0}\left(\tilde{K}_{rr}r'^2-\tilde{V}\right),\qquad \tilde{K}_{rr}=-(2\lambda^0T_{D2})^2r_0Me^{-\Phi},$$
$$\tilde{V}=r_0^{1/2}C\left(v_0^2-\bar{\Lambda}_-^2A^2-\bar{\Lambda}_+^2B^2-\Lambda_D^2D^2\right)e^{-\Phi},$$

where M, v_0^2, $\bar{\Lambda}_\pm^2$ and Λ_D^2 are defined in (5.2), (4.5) and (4.15) respectively.

[5]In this limit $v_\pm$ remain finite.

JHEP08(2006)029

The solution $\xi^2(r)$ can be obtained from (5.3) by the replacement

$$\Lambda_+^2 + \Lambda_-^2 \to \bar{\Lambda}_+^2 + \bar{\Lambda}_-^2 + 4\Lambda_D^2/3. \tag{5.14}$$

It is understood, that the solutions $r_{max} = r_1$ and r_2 of $r' = 0$ are also correspondingly changed (r_{min} remains the same). The explicit expressions for E and P_I can be obtained in the same way from (5.4). The computation of the conserved angular momenta P_θ and P_ϕ according to (3.37) gives

$$\begin{aligned}
P_\theta &= \left(\Lambda_0^\theta - \Lambda_0^\phi \sin\psi_1^0 \sin\theta_2^0\right) J_{A1}^D + \left(\Lambda_0^\theta + \Lambda_0^\phi \sin\psi_1^0 \sin\theta_2^0\right) J_{B1}^D,\\
P_\phi &= \left(\Lambda_0^\phi \sin\theta_2^0 - \Lambda_0^\theta \sin\psi_1^0\right) \sin\theta_2^0 J_{A1}^D + \left(\Lambda_0^\phi \sin\theta_2^0 + \Lambda_0^\theta \sin\psi_1^0\right) \sin\theta_2^0 J_{B1}^D\\
&\quad + \Lambda_0^\phi \cos^2\theta_2^0 J_{D1}^D,
\end{aligned}$$

where one obtains J_{A1}^D, J_{B1}^D from (5.6), (5.7) by the replacement (5.14), and

$$\begin{aligned}
J_{D1}^D &= 8\pi^2 T_{D2}\left[\frac{Ml^5}{(\bar{\Lambda}_+^2 + \bar{\Lambda}_-^2 + 4\Lambda_D^2/3)(3l - r_2)}\right]^{1/2} \times\\
&\left(1+\frac{\Delta r_1}{2l}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l}\right)^{2}\left(1+\frac{\Delta r_1}{4l}\right)^{1/2}\left(1+\frac{\Delta r_1}{6l}\right)^{-1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2} \times\\
&F_D^{(5)}\left(1/2; -1/2, -2, -1/2, 1/2, 1/2; 1;\right.\\
&\left.\frac{1}{1+\frac{2l}{\Delta r_1}}, \frac{1}{1+\frac{3l}{\Delta r_1}}, \frac{1}{1+\frac{4l}{\Delta r_1}}, \frac{1}{1+\frac{6l}{\Delta r_1}}, \frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right).
\end{aligned}$$

Taking $r_1 \to \infty$ in the above expressions, one obtains that in the semiclassical limit the following energy-charge relation holds

$$\frac{E^2\left(E^2 - \mathbf{P}^2\right)^2}{\mathbf{\Lambda}_1^2 E^2 - (\mathbf{\Lambda}_1.\mathbf{P})^2} = \frac{2^3}{3^5}(\pi^2 T_{D2})^2\left(P_\theta^2 + \frac{3P_\phi^2}{3-\cos^2\theta_2^0}\right).$$

Obviously, this is a generalization of the relation (5.8) and for $\theta_2^0 = \pi/2$ has the same form.

Let us see if another D2-brane embedding for the target space metric (4.2) is possible. It turns out that in this case such nontrivial solution exists if the non-diagonal part of the metric (4.2) is absent. Otherwise, we have too many conditions on the embedding parameters, which leads to vanishing kinetic term in the Lagrangian (3.27): $\tilde{K}_{rr} = 0$. That is why, we will consider the particular case $\psi_1^0 = 0$. Then, the other possible ansatz is

$$X^0 = \Lambda_0^0\xi^0, \quad X^I = \Lambda_0^I\xi^0, \quad r = r(\xi^2), \quad \theta_1 = \Lambda_1^\theta\xi^1 + \Lambda_2^\theta\xi^2, \quad \phi_2 = \Lambda_0^\phi\xi^0, \tag{5.15}$$

i.e., we have D2-brane extended in the radial direction r, wrapped along the angular coordinate θ_1 and rotating in the plane given by the angle ϕ_2. The embedding

$$X^0 = \Lambda_0^0\xi^0, \quad X^I = \Lambda_0^I\xi^0, \quad r = r(\xi^2), \quad \theta_1 = \Lambda_0^\theta\xi^0, \quad \phi_2 = \Lambda_1^\phi\xi^1 + \Lambda_2^\phi\xi^2$$

is also admissible, but it just interchanges the role of the angles θ_1 and ϕ_2.

For the ansatz (5.15), the Lagrangian (3.27) is given by

$$\mathcal{L}^A(\xi^2) = \frac{1}{4\lambda^0}\left(\tilde{K}_{rr}r'^2 - \tilde{V}\right), \quad \tilde{K}_{rr} = -(2\lambda^0 T_{D2})^2 r_0 (\Lambda_1^\theta)^2 \left(A^2 + B^2\right) e^{-\Phi},$$
$$\tilde{V} = r_0^{1/2} C\left[v_0^2 - \Lambda^2\left(A^2 + B^2\right) - \Lambda_D^2 D^2\right] e^{-\Phi},$$

where

$$\Lambda^2 = (\Lambda_0^\phi)^2 \sin^2\theta_2^0.$$

v_0^2 and Λ_D^2 are introduced in (4.5) and (4.15) respectively. The solutions of the equation $r' = 0$ determining the turning points of the periodic motion now are:

$$r_{min} = 3l, \quad r_{max} = r_1 = 3\sqrt{\frac{2v_0^2}{3\Lambda^2 + 2\Lambda_D^2}} = -r_2.$$

Replacing the above expressions for $\tilde{K}_{rr}$ and $\tilde{V}$ in (3.36), one obtains the solution:

$$\xi^2(r) = \frac{8}{3}\lambda^0 T_{D2}\Lambda_1^\theta \left[\frac{l(3l - w_+)(3l - w_-)}{\left(3\Lambda^2 + 2\Lambda_D^2\right)(3l - r_2)\,\Delta r_1}\right]^{1/2} (2\Delta r)^{3/4} \times$$
$$F_D^{(7)}\left(3/4; -1/4, -1/4, 1/4, -1/2, -1/2, 1/2, 1/2; 7/4;\right. \tag{5.16}$$
$$\left. -\frac{\Delta r}{2l}, -\frac{\Delta r}{4l}, -\frac{\Delta r}{6l}, -\frac{\Delta r}{3l - w_+}, -\frac{\Delta r}{3l - w_-}, -\frac{\Delta r}{3l - r_2}, \frac{\Delta r}{\Delta r_1}\right), \quad w_\pm = \pm\sqrt{3}l.$$

The computation of the conserved quantities E, P_I and $P_{\phi_2} \equiv P_\phi$, in accordance with (3.37), gives

$$\frac{E}{\Lambda_0^0} = \frac{P_I}{\Lambda_0^I} = 4\pi^2 T_{D2}\Lambda_1^\theta \left[\frac{l(3l - w_+)(3l - w_-)}{\left(3\Lambda^2 + 2\Lambda_D^2\right)(3l - r_2)}\right]^{1/2} \times \tag{5.17}$$
$$\left(1 + \frac{\Delta r_1}{2l}\right)^{1/2}\left(1 + \frac{\Delta r_1}{4l}\right)^{1/2}\left(1 + \frac{\Delta r_1}{6l}\right)^{-1/2} \times$$
$$\left(1 + \frac{\Delta r_1}{3l - w_+}\right)^{1/2}\left(1 + \frac{\Delta r_1}{3l - w_-}\right)^{1/2}\left(1 + \frac{\Delta r_1}{3l - r_2}\right)^{-1/2} \times$$
$$F_D^{(6)}\left(1/2; -1/2, -1/2, 1/2, -1/2, -1/2, 1/2; 1;\right.$$
$$\left.\frac{1}{1 + \frac{2l}{\Delta r_1}}, \frac{1}{1 + \frac{4l}{\Delta r_1}}, \frac{1}{1 + \frac{6l}{\Delta r_1}}, \frac{1}{1 + \frac{3l - w_+}{\Delta r_1}}, \frac{1}{1 + \frac{3l - w_-}{\Delta r_1}} \frac{1}{1 + \frac{3l - r_2}{\Delta r_1}}\right),$$

$$P_\phi = \sin^2\theta_2^0\left(J_{A2}^D + J_{B2}^D\right) + \cos^2\theta_2^0 J_{D2}^D,$$

where

$$J_{A2}^D = 4\pi^2 T_{D2}\Lambda_0^\phi \Lambda_1^\theta \left[\frac{l^5(3l - w_+)(3l - w_-)}{\left(3\Lambda^2 + 2\Lambda_D^2\right)(3l - r_2)}\right]^{1/2} \times$$

$$\left(1+\frac{\Delta r_1}{2l}\right)^{3/2}\left(1+\frac{\Delta r_1}{4l}\right)^{1/2}\left(1+\frac{\Delta r_1}{6l}\right)^{1/2}\times$$
$$\left(1+\frac{\Delta r_1}{3l-w_+}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-w_-}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\times$$
$$F_D^{(6)}\left(1/2;-3/2,-1/2,-1/2,-1/2,-1/2,1/2;1;\right.$$
$$\left.\frac{1}{1+\frac{2l}{\Delta r_1}},\frac{1}{1+\frac{4l}{\Delta r_1}},\frac{1}{1+\frac{6l}{\Delta r_1}},\frac{1}{1+\frac{3l-w+}{\Delta r_1}},\frac{1}{1+\frac{3l-w-}{\Delta r_1}}\frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right),$$

$$J_{B2}^D=\frac{2}{3}\pi^2 T_{D2}\Lambda_0^\phi\Lambda_1^\theta\left[\frac{l^3(3l-w_+)(3l-w_-)}{\left(3\Lambda^2+2\Lambda_D^2\right)(3l-r_2)}\right]^{1/2}\times$$
$$\Delta r_1\left(1+\frac{\Delta r_1}{2l}\right)^{1/2}\left(1+\frac{\Delta r_1}{4l}\right)^{3/2}\left(1+\frac{\Delta r_1}{6l}\right)^{-1/2}\times$$
$$\left(1+\frac{\Delta r_1}{3l-w_+}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-w_-}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\times$$
$$F_D^{(6)}\left(1/2;-1/2,-3/2,1/2,-1/2,-1/2,1/2;2;\right.$$
$$\left.\frac{1}{1+\frac{2l}{\Delta r_1}},\frac{1}{1+\frac{4l}{\Delta r_1}},\frac{1}{1+\frac{6l}{\Delta r_1}},\frac{1}{1+\frac{3l-w+}{\Delta r_1}},\frac{1}{1+\frac{3l-w-}{\Delta r_1}}\frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right),$$

$$J_{D2}^D=4\pi^2 T_{D2}\Lambda_0^\phi\Lambda_1^\theta\left[\frac{l^5(3l-w_+)(3l-w_-)}{\left(3\Lambda^2+2\Lambda_D^2\right)(3l-r_2)}\right]^{1/2}\times$$
$$\left(1+\frac{\Delta r_1}{2l}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l}\right)^{2}\left(1+\frac{\Delta r_1}{4l}\right)^{1/2}\left(1+\frac{\Delta r_1}{6l}\right)^{-1/2}\times$$
$$\left(1+\frac{\Delta r_1}{3l-w_+}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-w_-}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\times$$
$$F_D^{(7)}\left(1/2;-1/2,-2,-1/2,1/2,-1/2,-1/2,1/2;1;\right.$$
$$\left.\frac{1}{1+\frac{2l}{\Delta r_1}},\frac{1}{1+\frac{3l}{\Delta r_1}},\frac{1}{1+\frac{4l}{\Delta r_1}},\frac{1}{1+\frac{6l}{\Delta r_1}},\frac{1}{1+\frac{3l-w+}{\Delta r_1}},\frac{1}{1+\frac{3l-w-}{\Delta r_1}}\frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right).$$

Going to the semiclassical limit $r_1\to\infty$ in the above expressions for the conserved quantities, one obtains the following relation between them

$$E^2=\mathbf{P}^2+\frac{3^{7/3}}{2^{1/3}}\left(\frac{\pi T_{D2}\Lambda_1^\theta}{3-\cos^2\theta_2^0}\right)^{2/3}P_\phi^{4/3}. \tag{5.18}$$

This is a generalization of the energy-charge relation received in (5.12).

Our next task is to consider D2-branes rotating in the background (4.3). One admissible embedding is

$$X^0=\Lambda_0^0\xi^0+\frac{(\mathbf{\Lambda}_0.\mathbf{\Lambda}_1)}{\Lambda_0^0}\left(\xi^1+c\xi^2\right),\quad X^I=\Lambda_0^I\xi^0+\Lambda_1^I\left(\xi^1+c\xi^2\right), \tag{5.19}$$
$$r=r(\xi^2),\quad \phi_1=\Lambda_0^{\phi_1}\xi^0,\quad \phi_2=\Lambda_0^{\phi_2}\xi^0.$$

It is analogous to (5.1) and (5.13), but now the rotations are in the planes given by the angles ϕ_1 and ϕ_2.

The D2-brane Lagrangian (3.27) now reads

$$\mathcal{L}^A(\xi^2) = \frac{1}{4\lambda^0}\left(\tilde{K}_{rr}r'^2 - \tilde{V}\right), \quad \tilde{K}_{rr} = -(2\lambda^0 T_{D2})^2 r_0 M e^{-\Phi},$$
$$\tilde{V} = r_0^{1/2} C\left(v_0^2 - \tilde{\Lambda}_+^2 A^2 - \tilde{\Lambda}_-^2 B^2 - \tilde{\Lambda}_D^2 D^2\right) e^{-\Phi},$$

where M, v_0^2, $\tilde{\Lambda}_\pm^2$ and $\tilde{\Lambda}_D^2$ are defined in (5.2), (4.5) and (4.25) respectively. The values for $r_{max} = r_1$ and r_2, the solution $\xi^2(r)$, and the expressions for E, P_I, may be obtained from the corresponding quantities for the embedding (5.13) by the replacements $\bar{\Lambda}_\mp^2 \to \tilde{\Lambda}_\pm^2$, $\Lambda_D^2 \to \tilde{\Lambda}_D^2$. For the conserved angular momenta P_{ϕ_1} and P_{ϕ_2}, (3.37) gives

$$\begin{aligned} P_{\phi_1} &= \left(\Lambda_0^{\phi_1}\sin\theta_1^0 + \Lambda_0^{\phi_2}\cos\psi_1^0\sin\theta_2^0\right)\sin\theta_1^0 K_{A1}^D \\ &+ \left(\Lambda_0^{\phi_1}\sin\theta_1^0 - \Lambda_0^{\phi_2}\cos\psi_1^0\sin\theta_2^0\right)\sin\theta_1^0 K_{B1}^D \\ &+ \left(\Lambda_0^{\phi_1}\cos\theta_1^0 + \Lambda_0^{\phi_2}\cos\theta_2^0\right)\cos\theta_1^0 K_{D1}^D, \end{aligned} \tag{5.20}$$

$$\begin{aligned} P_{\phi_2} &= \left(\Lambda_0^{\phi_2}\sin\theta_2^0 + \Lambda_0^{\phi_1}\cos\psi_1^0\sin\theta_1^0\right)\sin\theta_2^0 K_{A1}^D \\ &+ \left(\Lambda_0^{\phi_2}\sin\theta_2^0 - \Lambda_0^{\phi_1}\cos\psi_1^0\sin\theta_1^0\right)\sin\theta_2^0 K_{B1}^D \\ &+ \left(\Lambda_0^{\phi_1}\cos\theta_1^0 + \Lambda_0^{\phi_2}\cos\theta_2^0\right)\cos\theta_2^0 K_{D1}^D, \end{aligned} \tag{5.21}$$

where K_{A1}^D, K_{B1}^D and K_{D1}^D can be obtained from J_{A1}^D, J_{B1}^D and J_{D1}^D through the above mentioned replacements.

The calculations show that in the semiclassical limit, the dependence of the energy on the conserved charges, for the present case, is given by the equality:

$$\begin{aligned} &\frac{E^2\left(E^2 - \mathbf{P}^2\right)^2}{\mathbf{\Lambda}_1^2 E^2 - (\mathbf{\Lambda}_1.\mathbf{P})^2} = \\ &\frac{2^3}{3^5}(\pi^2 T_{D2})^2 \frac{\left(3-\cos^2\theta_2^0\right)P_{\phi_1}^2 + \left(3-\cos^2\theta_1^0\right)P_{\phi_2}^2 - 4P_{\phi_1}P_{\phi_2}\cos\theta_1^0\cos\theta_2^0}{3-\cos^2\theta_1^0 - \cos^2\theta_2^0 - \cos^2\theta_1^0\cos^2\theta_2^0}. \end{aligned} \tag{5.22}$$

This is another generalization of the energy-charge relation $E \sim K^{1/2}$, and for $\theta_1^0 = \theta_2^0 = \pi/2$ has the same form as the relation in (5.8).

Finally, let us consider the other possible D2-brane embedding in the background (4.3). It turns out that such nontrivial embedding do exists only for $\theta_1^0 = \theta_2^0 \equiv \theta^0$, and is given by the ansatz

$$\begin{aligned} &X^0 = \Lambda_0^0\xi^0, \quad X^I = \Lambda_0^I\xi^0, \quad r = r(\xi^2), \\ &\phi_1 = \Lambda_0^\phi\xi^0 + \Lambda_1^\phi\xi^1 + \Lambda_2^\phi\xi^2, \quad \phi_2 = \Lambda_0^\phi\xi^0 - \Lambda_1^\phi\xi^1 - \Lambda_2^\phi\xi^2. \end{aligned} \tag{5.23}$$

It describes D2-brane configuration, which is analogous to the one in (5.9), but now the rotations are in the planes defined by the angles ϕ_1 and ϕ_2 instead of θ_1 and θ_2.

For this embedding, the Lagrangian (3.27) have the form

$$\mathcal{L}^A(\xi^2) = \frac{1}{4\lambda^0}\left(\tilde{K}_{rr}r'^2 - \tilde{V}\right), \quad \tilde{K}_{rr} = -(2\lambda^0 T_{D2})^2 r_0 \left(\hat{\Lambda}_{1-}^2 A^2 + \hat{\Lambda}_{1+}^2 B^2\right) e^{-\Phi},$$
$$\tilde{V} = r_0^{1/2} C \left(v_0^2 - \hat{\Lambda}_+^2 A^2 - \hat{\Lambda}_-^2 B^2 - \hat{\Lambda}_D^2 D^2\right) e^{-\Phi},$$

where v_0^2 is defined in (4.5) and

$$\hat{\Lambda}_{1\pm}^2 = 2(1 \pm \cos\psi_1^0)\sin^2\theta^0(\Lambda_1^\phi)^2,$$
$$\hat{\Lambda}_{\pm}^2 = 2(1 \pm \cos\psi_1^0)\sin^2\theta^0(\Lambda_0^\phi)^2,$$
$$\hat{\Lambda}_D^2 = 4\cos^2\theta^0(\Lambda_0^\phi)^2.$$

The constraint (3.33), $\tilde{K}_{rr}r'^2 + \tilde{V} = 0$, leads to the same solutions of the equation $r' = 0$, as for the case just considered, but in terms of the new parameters $\hat{\Lambda}_\pm^2$, $\hat{\Lambda}_D^2$.

In accordance with (3.36), one obtains

$$\xi^2(r) = \frac{8}{3}\lambda^0 T_{D2}\left[\frac{l\left(\hat{\Lambda}_{1+}^2 + \hat{\Lambda}_{1-}^2\right)(3l-u_+)(3l-u_-)}{3\left(\hat{\Lambda}_+^2 + \hat{\Lambda}_-^2 + 4\hat{\Lambda}_D^2/3\right)(3l-r_2)\,\Delta r_1}\right]^{1/2}(2\Delta r)^{3/4}\times$$
$$F_D^{(7)}\,(3/4; -1/4, -1/4, 1/4, -1/2, -1/2, 1/2, 1/2; 7/4;$$
$$-\frac{\Delta r}{2l}, -\frac{\Delta r}{4l}, -\frac{\Delta r}{6l}, -\frac{\Delta r}{3l-u_+}, -\frac{\Delta r}{3l-u_-}, -\frac{\Delta r}{3l-r_2}, \frac{\Delta r}{\Delta r_1}\Big),$$

where

$$u_\pm = l\left[\frac{\hat{\Lambda}_{1+}^2 - \hat{\Lambda}_{1-}^2}{\hat{\Lambda}_{1+}^2 + \hat{\Lambda}_{1-}^2} \pm \sqrt{3 + \left(\frac{\hat{\Lambda}_{1+}^2 - \hat{\Lambda}_{1-}^2}{\hat{\Lambda}_{1+}^2 + \hat{\Lambda}_{1-}^2}\right)^2}\,\right].$$

The computation of the conserved charges (3.37) results in

$$\frac{E}{\Lambda_0^0} = \frac{P_I}{\Lambda_0^I} = 4\pi^2 T_{D2}\left[\frac{l\left(\hat{\Lambda}_{1+}^2 + \hat{\Lambda}_{1-}^2\right)(3l-u_+)(3l-u_-)}{3\left(\hat{\Lambda}_+^2 + \hat{\Lambda}_-^2 + 4\hat{\Lambda}_D^2/3\right)(3l-r_2)}\right]^{1/2}\times \tag{5.24}$$
$$\left(1+\frac{\Delta r_1}{2l}\right)^{1/2}\left(1+\frac{\Delta r_1}{4l}\right)^{1/2}\left(1+\frac{\Delta r_1}{6l}\right)^{-1/2}\times$$
$$\left(1+\frac{\Delta r_1}{3l-u_+}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-u_-}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\times$$
$$F_D^{(6)}\,(1/2; -1/2, -1/2, 1/2, -1/2, -1/2, 1/2; 1;$$
$$\frac{1}{1+\frac{2l}{\Delta r_1}}, \frac{1}{1+\frac{4l}{\Delta r_1}}, \frac{1}{1+\frac{6l}{\Delta r_1}}, \frac{1}{1+\frac{3l-u+}{\Delta r_1}}, \frac{1}{1+\frac{3l-u-}{\Delta r_1}}\,\frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\Bigg),$$

$$P_\phi \equiv P_{\phi_1} = P_{\phi_2} =$$
$$\Lambda_0^\phi\left\{\sin^2\theta^0\left[\left(1+\cos\psi_1^0\right)K_{A2}^D + \left(1-\cos\psi_1^0\right)K_{B2}^D\right] + 2\cos^2\theta^0 K_{D2}^D\right\},$$

where

$$K_{A2}^{D}=4\pi^2 T_{D2}\left[\frac{l^5\left(\hat{\Lambda}_{1+}^2+\hat{\Lambda}_{1-}^2\right)(3l-u_+)(3l-u_-)}{3\left(\hat{\Lambda}_{+}^2+\hat{\Lambda}_{-}^2+4\hat{\Lambda}_{D}^2/3\right)(3l-r_2)}\right]^{1/2}\times$$
$$\left(1+\frac{\Delta r_1}{2l}\right)^{3/2}\left(1+\frac{\Delta r_1}{4l}\right)^{1/2}\left(1+\frac{\Delta r_1}{6l}\right)^{1/2}\times$$
$$\left(1+\frac{\Delta r_1}{3l-u_+}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-u_-}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\times$$
$$F_D^{(6)}\left(1/2;-3/2,-1/2,-1/2,-1/2,-1/2,1/2;1;\right.$$
$$\left.\frac{1}{1+\frac{2l}{\Delta r_1}},\frac{1}{1+\frac{4l}{\Delta r_1}},\frac{1}{1+\frac{6l}{\Delta r_1}},\frac{1}{1+\frac{3l-u+}{\Delta r_1}},\frac{1}{1+\frac{3l-u-}{\Delta r_1}}\frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right),$$

$$K_{B2}^{D}=2\pi^2 T_{D2}\left[\frac{l^3\left(\hat{\Lambda}_{1+}^2+\hat{\Lambda}_{1-}^2\right)(3l-u_+)(3l-u_-)}{3^3\left(\hat{\Lambda}_{+}^2+\hat{\Lambda}_{-}^2+4\hat{\Lambda}_{D}^2/3\right)(3l-r_2)}\right]^{1/2}\times$$
$$\Delta r_1\left(1+\frac{\Delta r_1}{2l}\right)^{1/2}\left(1+\frac{\Delta r_1}{4l}\right)^{3/2}\left(1+\frac{\Delta r_1}{6l}\right)^{-1/2}\times$$
$$\left(1+\frac{\Delta r_1}{3l-u_+}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-u_-}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\times$$
$$F_D^{(6)}\left(1/2;-1/2,-3/2,1/2,-1/2,-1/2,1/2;2;\right.$$
$$\left.\frac{1}{1+\frac{2l}{\Delta r_1}},\frac{1}{1+\frac{4l}{\Delta r_1}},\frac{1}{1+\frac{6l}{\Delta r_1}},\frac{1}{1+\frac{3l-u+}{\Delta r_1}},\frac{1}{1+\frac{3l-u-}{\Delta r_1}}\frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right),$$

$$K_{D2}^{D}=4\pi^2 T_{D2}\left[\frac{l^5\left(\hat{\Lambda}_{1+}^2+\hat{\Lambda}_{1-}^2\right)(3l-u_+)(3l-u_-)}{3\left(\hat{\Lambda}_{+}^2+\hat{\Lambda}_{-}^2+4\hat{\Lambda}_{D}^2/3\right)(3l-r_2)}\right]^{1/2}\times$$
$$\left(1+\frac{\Delta r_1}{2l}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l}\right)^{2}\left(1+\frac{\Delta r_1}{4l}\right)^{1/2}\left(1+\frac{\Delta r_1}{6l}\right)^{-1/2}\times$$
$$\left(1+\frac{\Delta r_1}{3l-u_+}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-u_-}\right)^{1/2}\left(1+\frac{\Delta r_1}{3l-r_2}\right)^{-1/2}\times$$
$$F_D^{(7)}\left(1/2;-1/2,-2,-1/2,1/2,-1/2,-1/2,1/2;1;\right.$$
$$\left.\frac{1}{1+\frac{2l}{\Delta r_1}},\frac{1}{1+\frac{3l}{\Delta r_1}},\frac{1}{1+\frac{4l}{\Delta r_1}},\frac{1}{1+\frac{6l}{\Delta r_1}},\frac{1}{1+\frac{3l-u+}{\Delta r_1}},\frac{1}{1+\frac{3l-u-}{\Delta r_1}}\frac{1}{1+\frac{3l-r_2}{\Delta r_1}}\right).$$

Taking the semiclassical limit in the above expressions for E, P_I and P_ϕ, which in the case under consideration corresponds to

$$r_{1,2}\to\pm 2\sqrt{\frac{3v_0^2}{\hat{\Lambda}_{+}^2+\hat{\Lambda}_{-}^2+4\hat{\Lambda}_{D}^2/3}}\to\infty,$$

we receive that the energy depends on P_I and P_ϕ as follows

$$E^2 = \mathbf{P}^2 + 3^{7/3}\left(\frac{2\pi T_{D2}\Lambda_1^\phi \sin\theta^0}{4-\sin^2\theta^0}\right)^{2/3} P_\phi^{4/3}. \tag{5.25}$$

This is another generalization of the energy-charge relation given in (5.12).

6. Comments and conclusions

In this paper, we considered rotating strings and D2-branes on type IIA background, which arises as dimensional reduction of M-theory on manifold of G_2 holonomy, dual to $\mathcal{N}=1$ gauge theory in four dimensions. We obtained exact solutions and explicit expressions for the energy and other momenta (charges), which are conserved due to the presence of background isometries. They were given in terms of the hypergeometric functions of many variables $F_D^{(n)}(a;b_1,\ldots,b_n;c;z_1,\ldots,z_n)$, where for the different cases considered, n varies from one to seven.

We investigated the semiclassical limit of the conserved quantities and received different types of relations between them. Our aim was to check if strings and D2-branes rotating in this ten dimensional type IIA background, can reproduce the energy-charge relations obtained in [5] and [11] for rotating M2-branes on G_2 manifolds. We found that the rotating strings can reproduce only one type of semiclassical behavior, exhibited by rotating M2-branes. Our results are the following

$$\begin{aligned}
&E^2 = \mathbf{P}^2 + 2\pi T\,(6r_0)^{1/2}\left(P_{\theta_1}^2 + P_{\theta_2}^2\right)^{1/2},\\
&E^2 = \mathbf{P}^2 + 2\pi T\,(6r_0)^{1/2}\left(P_\theta^2 + \frac{3P_\phi^2}{3-\cos^2\theta_2^0}\right)^{1/2},\\
&E^2 = \mathbf{P}^2 + 2\pi T(6r_0)^{1/2}\times\\
&\left[\frac{\left(3-\cos^2\theta_2^0\right)P_{\phi_1}^2 + \left(3-\cos^2\theta_1^0\right)P_{\phi_2}^2 - 4P_{\phi_1}P_{\phi_2}\cos\theta_1^0\cos\theta_2^0}{3-\cos^2\theta_1^0-\cos^2\theta_2^0-\cos^2\theta_1^0\cos^2\theta_2^0}\right]^{1/2}.
\end{aligned}$$

These equalities are generalizations of the $E \sim K^{1/2}$ behavior and correspond to the following M2-brane energy-charge relation [11]

$$\begin{aligned}
&\left\{E^2\left(E^2-\mathbf{P}^2\right) - (2\pi^2 T_{M2} l_{11}^3)^2\left\{(\mathbf{\Lambda}_1\times\mathbf{\Lambda}_2)^2 E^2 - [(\mathbf{\Lambda}_1\times\mathbf{\Lambda}_2)\times\mathbf{P}]^2\right\}\right\}^2\\
&-6(2\pi^2 T_{M2} l_{11}^3)^2 E^2\left[\mathbf{\Lambda}_1^2 E^2 - (\mathbf{\Lambda}_1.\mathbf{P})^2\right]\left(P_\theta^2 + P_{\tilde\theta}^2\right) = 0.
\end{aligned} \tag{6.1}$$

We also showed that the rotating D2-branes reproduce two types of the semiclassical energy-charge relations known for membranes in M-theory. The first type is represented by

$$\frac{E^2\left(E^2-\mathbf{P}^2\right)^2}{\mathbf{\Lambda}_1^2 E^2 - (\mathbf{\Lambda}_1.\mathbf{P})^2} = \frac{2^3}{3^5}(\pi^2 T_{D2})^2\left(P_{\theta_1}^2 + P_{\theta_2}^2\right),$$

$$
\begin{aligned}
&\frac{E^2\left(E^2-\mathbf{P}^2\right)^2}{\mathbf{\Lambda}_1^2E^2-(\mathbf{\Lambda}_1.\mathbf{P})^2}=\frac{2^3}{3^5}(\pi^2T_{D2})^2\left(P_\theta^2+\frac{3P_\phi^2}{3-\cos^2\theta_2^0}\right),\\
&\frac{E^2\left(E^2-\mathbf{P}^2\right)^2}{\mathbf{\Lambda}_1^2E^2-(\mathbf{\Lambda}_1.\mathbf{P})^2}=\\
&\frac{2^3}{3^5}(\pi^2T_{D2})^2\frac{\left(3-\cos^2\theta_2^0\right)P_{\phi_1}^2+\left(3-\cos^2\theta_1^0\right)P_{\phi_2}^2-4P_{\phi_1}P_{\phi_2}\cos\theta_1^0\cos\theta_2^0}{3-\cos^2\theta_1^0-\cos^2\theta_2^0-\cos^2\theta_1^0\cos^2\theta_2^0}.
\end{aligned}
$$

These are generalizations of the dependence $E\sim K^{1/2}$ and correspond to (6.1). For the second type, we received the equalities

$$
\begin{aligned}
E^2&=\mathbf{P}^2+3^{5/3}(2\pi T_{D2}\Lambda_1^\theta)^{2/3}P_\theta^{4/3},\\
E^2&=\mathbf{P}^2+\frac{3^{7/3}}{2^{1/3}}\left(\frac{\pi T_{D2}\Lambda_1^\theta}{3-\cos^2\theta_2^0}\right)^{2/3}P_\phi^{4/3},\\
E^2&=\mathbf{P}^2+3^{7/3}\left(\frac{2\pi T_{D2}\Lambda_1^\phi\sin\theta^0}{4-\sin^2\theta^0}\right)^{2/3}P_\phi^{4/3},
\end{aligned}
$$

which are generalizations of the dependence $E\sim K^{2/3}$ and correspond to [11]

$$
E^2=\mathbf{P}^2+3^{5/3}(2\pi T_{M2}l_{11}^3\Lambda_1^6)^{2/3}P_\theta^{4/3}.
$$

We were not able to obtain the other three types of semiclassical behavior discovered in [11] for M2-branes

$$
\begin{aligned}
&E^2=\mathbf{P}^2+\frac{9}{2l^2}P_+^2-(6\pi^2T_{M2}l_{11}^3\Lambda_1^-)^{2/3}P_+^{4/3},\\
&\left\{E^2\left[E^2-\mathbf{P}^2-(3/l)^2P_+^2\right]-(2\pi^2T_{M2}l_{11}^3)^2\left\{(\mathbf{\Lambda}_1\times\mathbf{\Lambda}_2)^2E^2-\left[(\mathbf{\Lambda}_1\times\mathbf{\Lambda}_2)\times\mathbf{P}\right]^2\right\}\right\}^2\\
&-2^7(3\pi T_{M2}l_{11}^3)^2E^2\left[\mathbf{\Lambda}_1^2E^2-(\mathbf{\Lambda}_1.\mathbf{P})^2\right]P_+^2=0,\\
&\left\{E^2\left[E^2-\mathbf{P}^2-(3/2l)^2P_+^2\right]-(2\pi^2T_{M2}l_{11}^3)^2\left\{(\mathbf{\Lambda}_1\times\mathbf{\Lambda}_2)^2E^2-\left[(\mathbf{\Lambda}_1\times\mathbf{\Lambda}_2)\times\mathbf{P}\right]^2\right\}\right\}^2\\
&-(6\pi^2T_{M2}l_{11}^3)^2E^2\left[\mathbf{\Lambda}_1^2E^2-(\mathbf{\Lambda}_1.\mathbf{P})^2\right]P_-^2=0,
\end{aligned}
$$

which generalize the relations

$$
E-K\sim K^{1/3},\qquad E-K\sim const,\qquad E\sim K_1+const\frac{K_2}{K_1}.
$$

One reason is that after the dimensional reduction from eleven to ten dimensions, the term in the background metric proportional to $C^2(r)$ disappears (compare (2.1) with (2.3)). Besides, we considered very restricted class of solutions, depending only on the radial background coordinate. However, these are just kind of technical reasons. To our opinion, the physical cause behind is that other types of M2-brane's semiclassical behavior should be reproduced in ten dimensions by more complex non-perturbative states like bound states of fundamental strings and D-branes. Support for this conjecture are the results obtained in [8], where such relation has been found for flat space-time. More precisely, starting with rotating membranes solutions in flat eleven dimensions, and compactifying on a circle and on a torus, the authors of [8] have been able to identify non-perturbative states of type IIA and type IIB superstring theory, which represent spinning bound states of D-branes and fundamenal strings.

We note that in considering the semiclassical limit (large charges), we take into account only the leading terms in the expressions for the conserved quantities. However, there is no problem to include the higher order terms. An example is given in (4.31).

For comparison, we now give two known results about the energy-charge relations, obtained in the semiclassical limit, for strings moving in other curved type IIA backgrounds[6].

Rotating strings in a warped $AdS_6 \times S^4$ geometry have been considered in [16]. The warped $AdS_6 \times S^4$ is vacuum solution of the massive type IIA supergravity, which is expected to be dual to an $\mathcal{N} = 2$, $D = 5$ super-conformal Yang-Mills theory. For large conserved charges, the following relation between them has been found

$$E - \frac{3}{2}J = c_1 + \frac{c_2}{J^5} + \cdots .$$

At the leading order, this relation is of the type $E - K \sim const$, and is reproduced by one of the M2-brane configurations described above, but not by the strings and D2-branes considered here.

Pulsating strings in the same warped $AdS_6 \times S^4$ background have been semiclassically quantized in [17] with the result

$$E^2 = (J + 7/3)(J + 4) + \text{quantum corrections},$$

which in the leading order gives the $E - K \sim const$ behavior once again.

It seems to us that an interesting task, which deserves to be investigated, is the semiclassical behavior of the strings and D2-branes in the γ-deformed [18] background (2.1), in order to see the difference with the results obtained here, and to estimate the role of the Kaluza-Klein modes, following the idea developed in [19], and applied for semiclassical strings in [20]. This problem is under investigation and we hope to report about some progress soon.

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A. Hypergeometric functions $F_D^{(n)}$

Here, we give some properties of the hypergeometric functions of many variables $F_D^{(n)}$ used in our calculations. By definition [21], for $|z_j| < 1$,

$$F_D^{(n)}(a; b_1, \ldots, b_n; c; z_1, \ldots, z_n) = \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{(a)_{k_1+\cdots+k_n}(b_1)_{k_1} \ldots (b_n)_{k_n}}{(c)_{k_1+\cdots+k_n}} \frac{z_1^{k_1} \ldots z_n^{k_n}}{k_1! \ldots k_n!},$$

[6]See also [15],where spinning and rotating closed string solutions in $AdS_5 \times T^{1,1}$ background have been found, and has been shown how these solutions can be mapped onto rotating closed strings embedded in configurations of intersecting branes in type IIA string theory.

where

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)},$$

and $\Gamma(z)$ is the Euler's Γ-function. In particular, $F_D^{(1)}(a;b;c;z) = {}_2F_1(a,b;c;z)$ is the Gauss' hypergeometric function, and $F_D^{(2)}(a;b_1,b_2;c;z_1,z_2) = F_1(a,b_1,b_2;c;z_1,z_2)$ is one of the hypergeometric functions of two variables.

1. $F_D^{(n)}(a;b_1,\dots,b_i,\dots,b_j,\dots,b_n;c;z_1,\dots,z_i,\dots,z_j,\dots,z_n) =$
$F_D^{(n)}(a;b_1,\dots,b_j,\dots,b_i,\dots,b_n;c;z_1,\dots,z_j,\dots,z_i,\dots,z_n)$.

2. $F_D^{(n)}(a;b_1,\dots,b_n;c;z_1,\dots,z_n) =$
$\prod_{i=1}^{n}(1-z_i)^{-b_i}\, F_D^{(n)}\left(c-a;b_1,\dots,b_n;c;\frac{z_1}{z_1-1},\dots,\frac{z_n}{z_n-1}\right)$.

3. $F_D^{(n)}(a;b_1,\dots,b_{i-1},b_i,b_{i+1},\dots,b_n;c;z_1,\dots,z_{i-1},1,z_{i+1},\dots,z_n) =$
$\frac{\Gamma(c)\Gamma(c-a-b_i)}{\Gamma(c-a)\Gamma(c-b_i)}F_D^{(n-1)}(a;b_1,\dots,b_{i-1},b_{i+1},\dots,b_n;c-b_i;z_1,\dots,z_{i-1},z_{i+1},\dots,z_n)$.

4. $F_D^{(n)}(a;b_1,\dots,b_{i-1},b_i,b_{i+1},\dots,b_n;c;z_1,\dots,z_{i-1},0,z_{i+1},\dots,z_n) =$
$F_D^{(n-1)}(a;b_1,\dots,b_{i-1},b_{i+1},\dots,b_n;c;z_1,\dots,z_{i-1},z_{i+1},\dots,z_n)$.

5. $F_D^{(n)}(a;b_1,\dots,b_{i-1},0,b_{i+1},\dots,b_n;c;z_1,\dots,z_{i-1},z_i,z_{i+1},\dots,z_n) =$
$F_D^{(n-1)}(a;b_1,\dots,b_{i-1},b_{i+1},\dots,b_n;c;z_1,\dots,z_{i-1},z_{i+1},\dots,z_n)$.

6. $F_D^{(n)}(a;b_1,\dots,b_i,\dots,b_j,\dots,b_n;c;z_1,\dots,z_i,\dots,z_i,\dots,z_n) =$
$F_D^{(n-1)}(a;b_1,\dots,b_i+b_j,\dots,b_n;c;z_1,\dots,z_i,\dots,z_n)$.

7. $F_D^{(2n+1)}(a;a-c+1,b_2,b_2,\dots,b_{2n},b_{2n};c;-1,z_2,-z_2\dots,z_{2n},-z_{2n}) =$
$\frac{\Gamma(a/2)\Gamma(c)}{2\Gamma(a)\Gamma(c-a/2)}F_D^{(n)}(a/2;b_2,\dots,b_{2n};c-a/2;z_2^2,\dots,z_{2n}^2)$.

8. $F_D^{(2n+1)}\,(c-a;a-c+1,b_2,b_2,\dots,b_{2n},b_{2n};c;$
$1/2,-\frac{z_2}{1-z_2},\frac{z_2}{1+z_2},\dots,-\frac{z_{2n}}{1-z_{2n}},\frac{z_{2n}}{1+z_{2n}}\Big) =$
$\frac{\Gamma(a/2)\Gamma(c)}{2^{c-a}\Gamma(a)\Gamma(c-a/2)}F_D^{(n)}\left(c-a;b_2,\dots,b_{2n};c-a/2;-\frac{z_2^2}{1-z_2^2},\dots,-\frac{z_{2n}^2}{1-z_{2n}^2}\right)$.

9. $F_D^{(2)}\,(a;b,b;c;z,-z) = {}_3F_2\left(\begin{matrix} a/2,(a+1)/2,b \\ c/2,(c+1)/2;z^2 \end{matrix}\right)$.

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JHEP08(2006)029